

On the sigma-model of deformed special geometry

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ABSTRACT

We discuss the deformed sigma-model that arises when considering four-dimensional $N = 2$ abelian vector multiplets in the presence of an arbitrary chiral background field. In addition, we allow for a class of deformations of special geometry by non-holomorphic terms. We analyze the geometry of the sigma-model in terms of intrinsic torsion classes. We show that, generically, the deformed geometry is non-Kähler. We illustrate our findings with an example. We also express the deformed sigma-model in terms of the Hesse potential that underlies the real formulation of special geometry.

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1 Introduction

As is well known, the Wilsonian Lagrangian of $N = 2$ abelian vector multiplets in four dimensions contains complex scalar fields that give rise to a non-linear sigma-model whose target manifold is a special-Kähler manifold. The geometry of the target manifold is called special geometry [1, 2]. In the holomorphic formulation of special geometry, the Wilsonian Lagrangian is encoded in a holomorphic function $F(X)$ that depends on complex scalar fields X that reside in the vector multiplets. In the coupling to supergravity, $F(X)$ is required to be homogeneous of degree two, and the Kähler potential of the target-space is expressed in terms of $F(X)$ as [3]

$$K(z, \bar{z}) = -\ln \left[\frac{i(\bar{X}^I F_I - \bar{F}_{\bar{I}} X^I)}{|X^0|^2} \right]. \quad (1.1)$$

Here the index I labels the vector multiplets (in supergravity it takes the values $I = 0, 1, \dots, n$), and F_I denotes $F_I = \partial F(X)/\partial X^I$. Due to the homogeneity property of $F(X)$, this Kähler potential depends only on the ‘special’ holomorphic coordinates $z^i = X^i/X^0$ and their complex conjugates, where $i = 1, \dots, n$.

Special geometry may also be formulated in terms of ‘special’ real coordinates [4]. In the case of rigid special geometry, the special-Kähler manifold is a Hessian manifold. The underlying Hesse potential is related by Legendre transformation to $F(X)$ [5]. The Hesse potential also plays a fundamental role in the real formulation of local special geometry [6, 7, 5], as emphasized recently in [8, 9, 10, 11, 12].

The abelian vector multiplets may be further coupled to (scalar) chiral multiplets that describe either additional dynamical fields or background fields [13, 14]. We will refer to these fields simply as chiral background fields in the following. They will be denoted by \hat{A} . The holomorphic function F will then also depend on \hat{A} , so that now $F(X, \hat{A})$. An example of a

chiral background is provided by the coupling of the abelian vector multiplets to the square of the Weyl multiplet [15]. In this case \hat{A} is constructed out of fields of the Weyl multiplet, and the Wilsonian Lagrangian will contain couplings of the vector multiplets to the square of the Weyl tensor. The presence of a chiral background deforms the non-linear sigma-model. When setting \hat{A} (or a rescaled version denoted by Υ , see below) to a constant value, this becomes a deformation parameter that affects the geometry of the target manifold. A concrete case is provided by considering the sigma-model in a space-time that asymptotes to $AdS_2 \times S^2$, with Υ set to the attractor value $\Upsilon = -64$. This is the situation encountered in the context of the quantum entropy function [16], where the scalar fields asymptote to constant values, while fluctuating in the interior of space-time [17, 18].

In order to pass from the Wilsonian effective action to the 1PI low-energy effective action, one needs to integrate over the massless modes of the model. In the context of $N = 2$ theories this induces non-holomorphic modifications in the gauge and gravitational couplings of the theory that, at the Wilsonian level, are encoded in the holomorphic function F . These modifications are crucial to ensure that a model has the expected duality symmetries. An early example thereof is provided by the computation of the moduli dependence of string loop corrections to gauge coupling constants in heterotic string compactifications [19]. These modifications, as well as considerations based on the duality invariance of black hole entropy formulae [20, 21], suggested that special geometry can be consistently modified by a class of non-holomorphic deformations, whereby the holomorphic function F that characterizes the Wilsonian action is replaced by a non-holomorphic function [8, 22, 10]

$$F(Y, \bar{Y}, \Upsilon, \bar{\Upsilon}) = F^{(0)}(Y) + 2i \Omega(Y, \bar{Y}, \Upsilon, \bar{\Upsilon}) , \quad (1.2)$$

where Ω denotes a real (in general non-harmonic) function. Here, Y^I and Υ denote rescaled fields X^I and \hat{A} which we give in (2.6) [23]. They are invariant under Weyl and $U(1)$ transformations, which constitute symmetries of the underlying superconformal approach to supergravity. The dependence on the chiral background is encoded in Ω , and the Wilsonian limit is recovered by taking Ω to be harmonic.

It was shown recently in [24] that the non-holomorphic deformations of special geometry described by (1.2) occur in a generic setting. Based on this observation, we will study the modified sigma-model that is obtained by taking the Wilsonian sigma-model, which is based on a holomorphic function $F(Y, \Upsilon)$, and replacing $F(Y, \Upsilon)$ by the non-holomorphic function (1.2). Although it is not known whether there exists a supersymmetric effective action based on (1.2), this modified sigma-model represents an extension of the Wilsonian sigma-model with symplectic covariance built into it. The latter is a key feature of $N = 2$ systems [3, 13]. In the following, whenever $\Omega \neq 0$, we will refer to the sigma-model as deformed sigma-model.

We introduce the deformed sigma-model in section 2. It is coupled to supergravity, and given in (2.14) in terms of the fields Y^I and Υ . We analyse its geometry in terms of intrinsic torsion classes following [25]. To do so, we first express the deformed sigma-model in terms of projective coordinates. As mentioned above, we work with rescaled fields Y^I and Υ , rather than X^I and \hat{A} . Then, in the presence of a chiral background, the set of projective

coordinates is given by $z^i = Y^i/Y^0$, $\Psi = \Upsilon/(Y^0)^w$, where w denotes the Weyl weight of the chiral background field \hat{A} . The deformed sigma-model may also depend on the phase of Y^0 , which we denote by $R = \bar{Y}^0/Y^0$. The phase of Y^0 differs from the phase of X^0 by a compensating field \bar{h} . This compensator ensures invariance under local $U(1)$ transformations. When $\Omega = 0$, it is well known that the sigma-model target-space metric does not depend on R , i.e. the target-space metric has an isometry associated with this angular variable. When $\Omega \neq 0$, however, we will show that this is not any longer the case and that the target-space metric is non-Kähler, in general. When written in projective coordinates, we find that the deformed sigma-model is given by (2.32). It takes a rather simple form. The first term is the one that is present in the rigid limit when decoupling supergravity. Its form represents a generalization of the one that enters in the extrinsic construction of special-Kähler manifolds given in [7]. The second term arises when coupling the deformed sigma-model to supergravity. We discuss the rigid limit in subsection 2.2.2.

The chiral background field Ψ constitutes an priori independent field, and its presence makes the target-space metric of the deformed sigma-model depend on additional directions associated with Ψ and $\bar{\Psi}$ (and hence also with R). We may, however, consider treating the chiral background as a deformation parameter. Since Υ is inert under symplectic transformations [13], whereas Ψ isn't, it is Υ that we treat as a deformation parameter of the sigma-model by setting it to a constant value. In doing so, Ψ ceases to be an independent field. It becomes expressed in terms of z^i, \bar{z}^i and R , and this in turn affects the geometry of the deformed target-space. This effect is already present at the Wilsonian level, as we discuss in subsection 2.2.1.

Focussing on the case when Υ is a constant parameter, we analyze the geometry of the deformed target-space in terms of intrinsic torsion classes. This is done in section 3. We restrict to a hypersurface $R = \text{constant}$ and take the number of vector multiplets to be specified by $n = 3$, in order to be able to use the analysis given in [25]. We work to first order in Ω or, equivalently, to first order in the deformation parameter Υ . We express the intrinsic torsion in terms of target-space metric coefficients that are of first-order in Υ . We find that the torsion class \mathcal{W}_1 vanishes, whereas \mathcal{W}_2 is in general non-vanishing. For a manifold to admit an integrable complex structure, both \mathcal{W}_1 and \mathcal{W}_2 have to vanish. Thus, generically, the hypersurface $R = \text{constant}$ does not admit an integrable complex structure. The intrinsic torsion analysis can, in principle, be extended to any order in Ω .

In section 4 we discuss an example based on an STU-model in the presence of a chiral background which we identify with the square of the Weyl multiplet. Before discussing its intrinsic torsion classes we perform a coordinate change that simplifies the deformed target-space metric. The coordinate change is the one discussed in [10]. This coordinate change is motivated by duality considerations, as follows. In the presence of a chiral background field, the usual transformation laws of the fields Y^I under S- and T-duality get modified by terms that depend on the chiral background. It is convenient to introduce new fields \tilde{Y}^I that under dualities transform in the usual way, when chiral background fields are absent. These new

fields can be defined as follows [10],

$$\begin{aligned}\mathrm{Re} \tilde{Y}^I &= \mathrm{Re} Y^I, \\ \mathrm{Re} F_I^{(0)}(\tilde{Y}) &= \mathrm{Re} F_I(Y, \bar{Y}, \Upsilon, \tilde{\Upsilon}).\end{aligned}\tag{1.3}$$

The duality transformations for the fields \tilde{Y}^I are independent of Υ and $\tilde{\Upsilon}$, whereas those of Y^I depend on the chiral background. Applying the change of coordinates (1.3) to the STU-model in question and performing the intrinsic torsion analysis, we find that at first order in the deformation parameter Υ the target manifold is almost-Kähler, with non-vanishing intrinsic torsion \mathcal{W}_2 (and \mathcal{W}_5).

In section 5 we return to the deformed sigma-model (2.14) in big moduli space. We introduce real coordinates and express the deformed sigma-model in terms of derivatives of a Hesse potential H that is obtained by Legendre transformation of the Lagrangian $\mathcal{L} = \mathrm{Im} F - \Omega$. As shown in [24], the combination \mathcal{L} arises naturally in the context of point-particle Lagrangians that depend on coordinates and velocities. The Lagrangian of such a system can be reformulated in terms of complex coordinates, and in these coordinates it equals \mathcal{L} . The Hesse potential H is then the Hamiltonian of this system. In the absence of a chiral background, the expressions we obtain reduce to those obtained recently in [12]. We conclude with a remark.

2 The sigma-model in the presence of a chiral background field

We consider the scalar field sigma-model that arises in four-dimensional $N = 2$ supergravity theories based on abelian vector multiplets coupled to supergravity in the presence of an arbitrary chiral background field [13, 14]. These theories can be conveniently described in terms of the superconformal multiplet calculus [26, 27, 3, 28]. At the Wilsonian level, the sigma-model is encoded in a holomorphic function F that is homogeneous of degree two. This function gets extended to the non-holomorphic function (1.2) when deforming special geometry by non-holomorphic terms [8, 22, 10] (see [24] for a review and applications). In this section, we discuss the implications of this modification for the sigma-model.

We begin with an analysis of the sigma-model in the presence of a chiral background field at the Wilsonian level. Subsequently we extend the discussion and include the non-holomorphic deformation of special geometry.

2.1 The sigma-model Lagrangian at the Wilsonian level

We consider the Wilsonian action describing the coupling of n abelian vector fields to supergravity in the presence of an arbitrary chiral background field. This action can be constructed in a transparent way by making use of the superconformal multiplet calculus, which incorporates the gauge symmetries of the $N = 2$ superconformal algebra. To obtain an action that is gauge equivalent to a Poincaré supergravity theory, two compensating multiplets need

to be coupled to conformal supergravity. One is a hyper multiplet, that will be omitted in the following, since it will not enter in the discussion of the sigma-model. The other is an abelian vector multiplet. We will therefore consider $n + 1$ abelian vector multiplets that will be labelled by an index $I = 0, 1, \dots, n$, with $I = 0$ referring to the compensating multiplet. Each of them contains a complex scalar field X^I with Weyl weight $w = 1$ and chiral weight $c = -1$. We will allow for the presence of an arbitrary chiral background superfield, whose lowest component field is a bosonic field \hat{A} that is complex and has Weyl weight w and chiral weight $c = -w$.

The coupling of the vector multiplets to conformal supergravity is encoded in a holomorphic function F that is homogeneous of degree two. In the presence of the chiral background, F depends on the complex scalar fields X^I as well as on \hat{A} , so that the homogeneity condition takes the form

$$F(\lambda X, \lambda^w \hat{A}) = \lambda^2 F(X, \hat{A}) , \quad (2.1)$$

where $\lambda \in \mathbb{C} \setminus \{0\}$. Therefore the function $F(X, \hat{A})$ satisfies the relation

$$2F = X^I F_I + w \hat{A} F_{\hat{A}} , \quad (2.2)$$

where F_I and $F_{\hat{A}}$ denote the derivatives of $F(X, \hat{A})$ with respect to X^I and \hat{A} , respectively.

As is well known, the field equations of the vector multiplets are subject to equivalence transformations corresponding to electric/magnetic duality transformations. They act as $\text{Sp}(2n+2; \mathbb{R})$ linear transformations on the $(2n+2)$ -component vector $(X^I, F_I(X, \hat{A}))$. While the background field \hat{A} is inert under duality transformations, it nevertheless enters in the explicit form of the transformations.

We introduce the following expressions [23],

$$\begin{aligned} e^{-\mathcal{K}} &= i \left(\bar{X}^I F_I(X, \hat{A}) - \bar{F}_{\bar{I}}(\bar{X}, \bar{\hat{A}}) X^I \right) , \\ \mathcal{A}_\mu &= \frac{1}{2} e^{\mathcal{K}} \left(\bar{X}^I \overleftrightarrow{\mathcal{D}}_\mu F_I - \bar{F}_{\bar{I}} \overleftrightarrow{\mathcal{D}}_\mu X^I \right) , \end{aligned} \quad (2.3)$$

as well as

$$N_{IJ} = -i (F_{IJ} - \bar{F}_{\bar{I}\bar{J}}) , \quad (2.4)$$

where $F_{IJ} = \partial^2 F(X, \hat{A}) / \partial X^I \partial X^J$. Observe that $e^{-\mathcal{K}}$ has weights $w = 2, c = 0$, while N_{IJ} has weights $w = c = 0$. The covariant derivative \mathcal{D}_μ is covariant with respect to chiral $U(1)$ transformations and dilations, and the associated gauge connections are A_μ and b_μ , respectively. Note that $\overleftrightarrow{\mathcal{D}}_\mu = \mathcal{D}_\mu - \overleftarrow{\mathcal{D}}_\mu$. Using (2.3), we infer the following expression for the $U(1)$ connection A_μ ,

$$A_\mu = -\frac{1}{2} e^{\mathcal{K}} \left(\bar{X}^I \overleftrightarrow{\mathcal{D}}_\mu F_I - \bar{F}_{\bar{I}} \overleftrightarrow{\mathcal{D}}_\mu X^I \right) + \mathcal{A}_\mu . \quad (2.5)$$

The combination \mathcal{A}_μ is $U(1)$ -invariant, and it vanishes in the absence of a chiral background. In the presence of a chiral background, \mathcal{A}_μ takes a complicated form that involves the quantity $F_{\hat{A}} = \partial F / \partial \hat{A}$ and derivatives thereof. In the case that the chiral background superfield is taken to be the square of the Weyl superfield, the expression for \mathcal{A}_μ can be found in [23].

Next, we introduce the rescaled Weyl and $U(1)$ invariant fields [23]

$$\begin{aligned} Y^I &= e^{\mathcal{K}/2} \bar{h} X^I, \\ \Upsilon &= \left(e^{\mathcal{K}/2} \bar{h} \right)^w \hat{A}, \end{aligned} \quad (2.6)$$

where h denotes a phase factor which transforms under $U(1)$ with the same weight as the fields X^I . Using the homogeneity property (2.1), we obtain $F(Y, \Upsilon) = e^{\mathcal{K}} \bar{h}^2 F(X, \hat{A})$ as well as

$$i \left(\bar{Y}^I F_I(Y, \Upsilon) - \bar{F}_I(\bar{Y}, \bar{\Upsilon}) Y^I \right) = 1, \quad (2.7)$$

where $F_I(Y, \Upsilon) = \partial F(Y, \Upsilon) / \partial Y^I$ (and similarly $F_{\Upsilon} = \partial F(Y, \Upsilon) / \partial \Upsilon$). Expressing (2.5) in terms of the rescaled fields (2.6) yields the $U(1)$ -invariant combination

$$a_{\mu} \equiv A_{\mu} - i \partial_{\mu} \ln h = -\frac{1}{2} \left(\bar{Y}^I \overleftrightarrow{\partial}_{\mu} F_I(Y, \Upsilon) - \bar{F}_I(\bar{Y}, \bar{\Upsilon}) \overleftrightarrow{\partial}_{\mu} Y^I \right) + \mathcal{A}_{\mu}. \quad (2.8)$$

We express $\mathcal{D}_{\mu} X^I$ in terms of the rescaled fields (2.6). We use the invariance under special conformal transformations to set $b_{\mu} = 0$, and we obtain

$$e^{\mathcal{K}/2} \bar{h} \mathcal{D}_{\mu} X^I = \partial_{\mu} Y^I + i a_{\mu} Y^I. \quad (2.9)$$

Now we turn to the kinetic term for the scalar fields X^I in the Wilsonian Lagrangian,

$$L = i \left(\mathcal{D}^{\mu} F_I(X, \hat{A}) \mathcal{D}_{\mu} \bar{X}^I - \mathcal{D}^{\mu} \bar{F}_I(\bar{X}, \bar{\hat{A}}) \mathcal{D}_{\mu} X^I \right), \quad (2.10)$$

which we rewrite in terms of the rescaled fields (2.6),

$$L = e^{-\mathcal{K}} \left[i \left(\partial^{\mu} F_I(Y, \Upsilon) \partial_{\mu} \bar{Y}^I - \partial^{\mu} \bar{F}_I(\bar{Y}, \bar{\Upsilon}) \partial_{\mu} Y^I \right) - (a^{\mu} - \mathcal{A}^{\mu}) (a_{\mu} - \mathcal{A}_{\mu}) + \mathcal{A}^{\mu} \mathcal{A}_{\mu} \right]. \quad (2.11)$$

Observe that the combination $a_{\mu} - \mathcal{A}_{\mu}$ and \mathcal{A}_{μ} do not couple to one another. The combination $a_{\mu} - \mathcal{A}_{\mu}$, which is given in (2.8), is determined in terms of $(Y^I, F_I(Y, \Upsilon))$, and is non-vanishing in the absence of a chiral background. The term $\mathcal{A}^{\mu} \mathcal{A}_{\mu}$, on the other hand, vanishes when switching off the chiral background. When the chiral background is identified with the square of the Weyl tensor, this term has a structure that is distinct from those of the other terms in (2.11). For instance, $\mathcal{A}^{\mu} \mathcal{A}_{\mu}$ contains a term proportional to $T^{+} T^{-} T^{+} T^{-}$, where T^{\pm} refers to (anti)self-dual Lorentz tensors. The other terms in (2.11) do not contain such a term, since they are constructed out of derivatives of the vector $(Y^I, F_I(Y, \Upsilon))$. Also note that $\mathcal{A}^{\mu} \mathcal{A}_{\mu}$ is of order $\mathcal{O}(F_{\Upsilon}^2)$ and higher, and can thus be neglected when working to first order in F_{Υ} . Thus, in the following, we split the Wilsonian Lagrangian into two pieces, $L_{\sigma}^{\text{Wils}} + L_{\text{grav}}$, where L_{σ}^{Wils} denotes the sigma-model Lagrangian that we will be focussing on in the following,

$$L_{\sigma}^{\text{Wils}} = e^{-\mathcal{K}} \left[i \left(\partial^{\mu} F_I(Y, \Upsilon) \partial_{\mu} \bar{Y}^I - \partial^{\mu} \bar{F}_I(\bar{Y}, \bar{\Upsilon}) \partial_{\mu} Y^I \right) - (a^{\mu} - \mathcal{A}^{\mu}) (a_{\mu} - \mathcal{A}_{\mu}) \right], \quad (2.12)$$

while L_{grav} denotes the remaining part of the supergravity Lagrangian, which includes the term $\mathcal{A}^{\mu} \mathcal{A}_{\mu}$. The overall factor $e^{-\mathcal{K}}$ appearing in (2.12) may be absorbed by a rescaling of the space-time metric or, equivalently, set to a constant value by using the freedom under dilations. In the following, we will thus consider (2.12) with $e^{-\mathcal{K}} = 1$.

The sigma-model Lagrangian (2.12), which is entirely constructed out of derivatives of the vector $(Y^I, F_I(Y, \Upsilon))$, is manifestly invariant under symplectic transformations of $(Y^I, F_I(Y, \Upsilon))$, which leave Υ inert. We now deform this Wilsonian sigma-model Lagrangian by allowing for non-holomorphic deformations that are incorporated into F by performing the extension (1.2). The function $F(Y, \bar{Y}, \Upsilon, \bar{\Upsilon})$, which is homogeneous of degree two, can be decomposed into a holomorphic background independent piece $F^{(0)}(Y)$ and a real function $\Omega(Y, \bar{Y}, \Upsilon, \bar{\Upsilon})$ that encodes the background dependence. This decomposition is defined up to an anti-holomorphic function, and the associated equivalence transformation takes the form [24]

$$F^{(0)}(Y) \rightarrow F^{(0)}(Y) + g(Y, \Upsilon) \quad , \quad \Omega \rightarrow \Omega - \text{Im } g(Y, \Upsilon) \quad , \quad (2.13)$$

resulting in $F(Y, \bar{Y}, \Upsilon, \bar{\Upsilon}) \rightarrow F(Y, \bar{Y}, \Upsilon, \bar{\Upsilon}) + \bar{g}(\bar{Y}, \bar{\Upsilon})$. The latter leaves the vector $(Y^I, F_I(Y, \bar{Y}, \Upsilon, \bar{\Upsilon}))$ unaffected. Electric/magnetic duality now acts as an $\text{Sp}(2n+2, \mathbb{R})$ -transformation on this vector.

Thus, in the presence of non-holomorphic terms, the deformed sigma-model Lagrangian will be specified by

$$L_\sigma = i \left(\partial^\mu F_I(Y, \bar{Y}, \Upsilon, \bar{\Upsilon}) \partial_\mu \bar{Y}^I - \partial^\mu \bar{F}_{\bar{I}}(Y, \bar{Y}, \Upsilon, \bar{\Upsilon}) \partial_\mu Y^I \right) - \mathcal{A}^\mu \mathcal{A}_\mu \quad , \quad (2.14)$$

where

$$\mathcal{A}_\mu = -\frac{1}{2} \left(\bar{Y}^I \overleftrightarrow{\partial}_\mu F_I(Y, \bar{Y}, \Upsilon, \bar{\Upsilon}) - \bar{F}_{\bar{I}}(Y, \bar{Y}, \Upsilon, \bar{\Upsilon}) \overleftrightarrow{\partial}_\mu Y^I \right) \quad , \quad (2.15)$$

with the Y^I satisfying the relation

$$i \left(\bar{Y}^I F_I(Y, \bar{Y}, \Upsilon, \bar{\Upsilon}) - \bar{F}_{\bar{I}}(Y, \bar{Y}, \Upsilon, \bar{\Upsilon}) Y^I \right) = 1 \quad . \quad (2.16)$$

The Wilsonian limit is recovered by taking Ω to be harmonic,

$$\Omega_{\text{Wils}}(Y, \bar{Y}, \Upsilon, \bar{\Upsilon}) = f(Y, \Upsilon) + \bar{f}(\bar{Y}, \bar{\Upsilon}) \quad , \quad (2.17)$$

which, through the equivalence transformation (2.13), results in $F(Y, \Upsilon) = F^{(0)}(Y) + 2if(Y, \Upsilon)$. In the next section, we turn to the evaluation of the deformed sigma-model Lagrangian (2.14).

2.2 The sigma-model Lagrangian in the presence of non-holomorphic terms

The deformed sigma-model L_σ introduced above is defined in terms of fields (Y^I, Υ) . In the following, we express this sigma-model in terms of projective coordinates given by

$$\begin{aligned} z^i &= \frac{Y^i}{Y^0} \quad , \quad i = 1, \dots, n \quad , \\ \Psi &= \frac{\Upsilon}{(Y^0)^w} \quad , \\ R &= \frac{\bar{Y}^0}{Y^0} \quad . \end{aligned} \quad (2.18)$$

Here $R = 1/\bar{R}$ denotes the phase of $(Y^0)^{-2}$. Observe that all these coordinates (including Ψ) transform under symplectic transformations of (Y^I, F_I) . The norm of Y^0 is expressed in terms

of these projective coordinates using the relation (2.16), as follows. Using the homogeneity of $F(Y, \bar{Y}, \Upsilon, \bar{\Upsilon})$, we obtain

$$F(Y, \bar{Y}, \Upsilon, \bar{\Upsilon}) = (Y^0)^2 \mathcal{F}(z, \bar{z}, \Psi, \bar{\Psi}, R) \quad (2.19)$$

with

$$\begin{aligned} \mathcal{F}(z, \bar{z}, \Psi, \bar{\Psi}, R) &= \mathcal{F}^{(0)}(z) + 2i \Omega(z, \bar{z}, \Psi, \bar{\Psi}, R) , \\ \bar{\mathcal{F}}(z, \bar{z}, \Psi, \bar{\Psi}, \bar{R}) &= \bar{\mathcal{F}}^{(0)}(\bar{z}) - 2i \Omega(\bar{z}, z, \bar{\Psi}, \Psi, \bar{R}) , \end{aligned} \quad (2.20)$$

where $\Omega(z, \bar{z}, \Psi, \bar{\Psi}, R)$ is not any longer a real function due to its dependence on the phase R . Using the expressions for the first-order derivatives of $F(Y, \bar{Y}, \Upsilon, \bar{\Upsilon})$ given in (A.2) and inserting these expressions into (2.16) yields

$$|Y^0|^2 = -i \Sigma^{-1} \quad (2.21)$$

with Σ given by

$$\Sigma(z, \bar{z}, \Psi, \bar{\Psi}, R) = 2(\mathcal{F} - \bar{\mathcal{F}}) - (z^i - \bar{z}^i)(\mathcal{F}_i + \bar{\mathcal{F}}_{\bar{i}}) - w(\Psi \mathcal{F}_\Psi - \bar{\Psi} \bar{\mathcal{F}}_{\bar{\Psi}}) - (R \mathcal{F}_R - \bar{R} \bar{\mathcal{F}}_{\bar{R}}) . \quad (2.22)$$

Observe that Σ satisfies $\Sigma = -\bar{\Sigma}$, and that it can be expressed as $\Sigma = \Delta - \bar{\Delta}$ with

$$\Delta(z, \bar{z}, \Psi, \bar{\Psi}, R) = 2\mathcal{F} - (z^i - \bar{z}^i)\mathcal{F}_i - w\Psi \mathcal{F}_\Psi - R \mathcal{F}_R . \quad (2.23)$$

Now we express \mathcal{A}^μ given in (2.15) in terms of the coordinates (2.18). First, using (2.16), we note that \mathcal{A}_μ can be written as

$$\mathcal{A}_\mu = \bar{F}_{\bar{I}} \partial_\mu Y^I - \bar{Y}^I \partial_\mu F_I = F_I \partial_\mu \bar{Y}^I - Y^I \partial_\mu \bar{F}_{\bar{I}} . \quad (2.24)$$

Subsequently we obtain

$$\begin{aligned} \mathcal{A}_\mu &= -\Sigma \bar{Y}^0 \partial_\mu Y^0 - |Y^0|^2 \mathfrak{a}_\mu = -\bar{\Sigma} Y^0 \partial_\mu \bar{Y}^0 - |Y^0|^2 \bar{\mathfrak{a}}_\mu , \\ \mathfrak{a}_\mu &= -\bar{\mathcal{F}}_{\bar{i}} \partial_\mu z^i - \mathcal{F}_i \partial_\mu \bar{z}^i + \partial_\mu \Delta \\ &= (\Delta_i - \bar{\mathcal{F}}_{\bar{i}}) \partial_\mu z^i + (\Delta_{\bar{i}} - \mathcal{F}_i) \partial_\mu \bar{z}^i + \Delta_\Psi \partial_\mu \Psi + \Delta_{\bar{\Psi}} \partial_\mu \bar{\Psi} + \Delta_R \partial_\mu R , \end{aligned} \quad (2.25)$$

as well as

$$\begin{aligned} \mathcal{A}^\mu \mathcal{A}_\mu &= |\Sigma|^2 |Y^0|^2 \partial_\mu Y^0 \partial^\mu \bar{Y}^0 + |Y^0|^4 \mathfrak{a}_\mu \bar{\mathfrak{a}}^\mu , \\ &+ \Sigma |Y^0|^2 \bar{Y}^0 \partial_\mu Y^0 \bar{\mathfrak{a}}^\mu + \bar{\Sigma} |Y^0|^2 Y^0 \partial_\mu \bar{Y}^0 \mathfrak{a}^\mu . \end{aligned} \quad (2.26)$$

The field strength of \mathcal{A}_μ reads

$$\begin{aligned} \mathcal{F}_{\mu\nu} &= \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu \\ &= \partial_\mu F_I \partial_\nu \bar{Y}^I - \partial_\mu Y^I \partial_\nu \bar{F}_{\bar{I}} - (\mu \leftrightarrow \nu) \\ &= \frac{i}{\Sigma} \left[\partial_\mu \mathfrak{a}_\nu - \partial_\nu \mathfrak{a}_\mu - \frac{1}{\Sigma} (\partial_\mu \Sigma \mathfrak{a}_\nu - \partial_\nu \Sigma \mathfrak{a}_\mu) \right] , \end{aligned} \quad (2.27)$$

where we used (2.21).

We proceed to express the sigma-model Lagrangian (2.14) in terms of projective coordinates. Using the expressions for the second-order derivatives of F given in (A.3) we obtain,

$$\begin{aligned}
F_{00} + F_{0i}(z^i + \bar{z}^i) + F_{ij} z^i \bar{z}^j + w\Psi (Y^0)^{w-1} (F_{0\Upsilon} + F_{i\Upsilon} \bar{z}^i) &= \Delta - R \Delta_R , \\
F_{0i} - \bar{F}_{0\bar{i}} + (F_{ij} - \bar{F}_{i\bar{j}}) z^j + F_{i\Upsilon} w\Psi (Y^0)^{w-1} &= \mathcal{F}_i - \bar{\Delta}_{\bar{i}} - R \mathcal{F}_{iR} , \\
(F_{0\Upsilon} + F_{i\Upsilon} \bar{z}^i) (Y^0)^{w-1} &= \Delta_\Psi , \\
F_{0\bar{0}} + F_{0\bar{i}} \bar{z}^i + F_{i\bar{0}} \bar{z}^i + F_{i\bar{j}} \bar{z}^i \bar{z}^j + F_{0\bar{\Upsilon}} w\bar{\Psi} (\bar{Y}^0)^{w-1} + F_{i\bar{\Upsilon}} \bar{z}^i w\bar{\Psi} (\bar{Y}^0)^{w-1} &= \Delta_{\bar{R}} , \\
F_{0\bar{i}} + F_{i\bar{0}} + (F_{j\bar{i}} + F_{i\bar{j}}) \bar{z}^j + F_{i\bar{\Upsilon}} w\bar{\Psi} (\bar{Y}^0)^{w-1} &= \mathcal{F}_{iR} + \bar{R} (\Delta_{\bar{i}} - \mathcal{F}_{\bar{i}}) , \\
(F_{0\bar{\Upsilon}} + F_{i\bar{\Upsilon}} \bar{z}^i) (\bar{Y}^0)^{w-1} &= \Delta_{\bar{\Psi}} .
\end{aligned} \tag{2.28}$$

Using these we get

$$\begin{aligned}
i (\partial_\mu F_I \partial^\mu \bar{Y}^I - \partial_\mu \bar{F}_{\bar{I}} \partial^\mu Y^{\bar{I}}) &= i \left[\partial_\mu Y^0 \partial^\mu \bar{Y}^0 (\Sigma - R \Delta_R + \bar{R} \Delta_{\bar{R}}) \right. \\
&\quad + \partial_\mu Y^0 \partial^\mu \bar{z}^i \bar{Y}^0 (\mathcal{F}_i - \bar{\Delta}_{\bar{i}} - R \mathcal{F}_{iR}) \\
&\quad - \partial_\mu \bar{Y}^0 \partial^\mu z^i Y^0 (\bar{\mathcal{F}}_{\bar{i}} - \Delta_i - \bar{R} \bar{\mathcal{F}}_{\bar{i}\bar{R}}) \\
&\quad + \partial_\mu z^i \partial^\mu \bar{z}^j |Y^0|^2 (\mathcal{F}_{ij} - \bar{\mathcal{F}}_{i\bar{j}}) \\
&\quad + \partial_\mu \Psi \partial^\mu \bar{Y}^0 Y^0 \Delta_\Psi - \partial_\mu \bar{\Psi} \partial^\mu Y^0 \bar{Y}^0 \bar{\Delta}_{\bar{\Psi}} \\
&\quad \left. + \partial_\mu \Psi \partial^\mu \bar{z}^i |Y^0|^2 \mathcal{F}_{i\Psi} - \partial_\mu \bar{\Psi} \partial^\mu z^i |Y^0|^2 \bar{\mathcal{F}}_{\bar{i}\bar{\Psi}} \right] \\
&\quad + i \left[\partial_\mu \bar{Y}^0 \partial^\mu \bar{Y}^0 \Delta_R \right. \\
&\quad + \partial_\mu \bar{Y}^0 \partial^\mu \bar{z}^i \bar{Y}^0 (\mathcal{F}_{iR} + \bar{R} (\Delta_{\bar{i}} - \mathcal{F}_{\bar{i}})) \\
&\quad + \partial_\mu \bar{z}^i \partial^\mu \bar{z}^j |Y^0|^2 \mathcal{F}_{i\bar{j}} \\
&\quad + \partial_\mu \bar{Y}^0 \partial^\mu \bar{\Psi} Y^0 \Delta_{\bar{\Psi}} \\
&\quad \left. + \partial_\mu \bar{z}^i \partial^\mu \bar{\Psi} |Y^0|^2 \mathcal{F}_{i\bar{\Psi}} - \text{c.c.} \right] .
\end{aligned} \tag{2.29}$$

Next, using the relation

$$\partial_\mu R = \frac{\partial_\mu \bar{Y}^0}{Y^0} - R \frac{\partial_\mu Y^0}{Y^0} , \tag{2.30}$$

as well as (2.21) with $\Sigma = -\bar{\Sigma}$, we obtain

$$\begin{aligned}
L_\sigma &= i |Y^0|^2 \left[\partial_\mu z^i \partial^\mu \bar{z}^j (\mathcal{F}_{ij} - \bar{\mathcal{F}}_{i\bar{j}}) \right. \\
&\quad + \partial_\mu \Psi \partial^\mu \bar{z}^i \mathcal{F}_{i\Psi} - \partial_\mu \bar{\Psi} \partial^\mu z^i \bar{\mathcal{F}}_{\bar{i}\bar{\Psi}} \\
&\quad + \partial_\mu R \partial^\mu \bar{z}^i \mathcal{F}_{iR} - \partial_\mu \bar{R} \partial^\mu z^i \bar{\mathcal{F}}_{\bar{i}\bar{R}} \\
&\quad \left. + \left(\partial_\mu \bar{z}^i \partial^\mu \bar{z}^j \mathcal{F}_{i\bar{j}} + \partial_\mu \bar{z}^i \partial^\mu \bar{\Psi} \mathcal{F}_{i\bar{\Psi}} - \text{c.c.} \right) \right] - |Y^0|^4 \mathfrak{a}_\mu \bar{\mathfrak{a}}^\mu .
\end{aligned} \tag{2.31}$$

Introducing $\rho^M = (z^i, \bar{z}^i, \Psi, \bar{\Psi}, R)$, this can also be written as

$$L_\sigma = i |Y^0|^2 \left[\partial_\mu \rho^M \partial^\mu \bar{z}^j \mathcal{F}_{Mj} - \text{c.c.} \right] - |Y^0|^4 \mathfrak{a}_\mu \bar{\mathfrak{a}}^\mu , \tag{2.32}$$

with \mathfrak{a}_μ given in (2.25). Thus we find that the deformed sigma-model Lagrangian takes a rather simple form, with the first term only depending on double derivatives of $\mathcal{F}(z, \bar{z}, \Psi, \bar{\Psi}, R)$. The Lagrangian (2.32) describes the coupling of the deformed sigma-model to supergravity. In the rigid limit, i.e. when decoupling supergravity in which case $\mathfrak{a}_\mu = 0$, the deformed sigma-model Lagrangian yields a generalization of the extrinsic construction of special-Kähler manifolds given in [7]. We will turn to this construction in the last subsection.

Note also that the Lagrangian (2.32) depends on the phase of Y^0 , i.e. on R . This means that when coupling the deformed sigma-model to supergravity, the associated target-space line element will depend on this coordinate, in general. This dependence drops out in the Wilsonian limit when taking (z^i, Ψ) to be independent fields, as we will now show.

2.2.1 The Wilsonian limit

In the Wilsonian limit (2.17), we have

$$\begin{aligned} F_{\text{Wils}}(Y, \bar{Y}, \Upsilon, \bar{\Upsilon}) &= (Y^0)^2 \mathcal{F}_{\text{Wils}}(z, \bar{z}, \Psi, \bar{\Psi}, R) , \\ \mathcal{F}_{\text{Wils}}(z, \bar{z}, \Psi, \bar{\Psi}, R) &= \mathcal{F}^H(z, \Psi) + 2i R^2 \bar{f}(\bar{z}, \bar{\Psi}) , \end{aligned} \quad (2.33)$$

where \mathcal{F}^H denotes the holomorphic part of $\mathcal{F}_{\text{Wils}}$,

$$\mathcal{F}^H(z, \Psi) = \mathcal{F}^{(0)}(z) + 2i f(z, \Psi) . \quad (2.34)$$

Inserting (2.33) into (2.22) results in

$$\Sigma^{\text{Wils}} = 2 (\mathcal{F}^H - \bar{\mathcal{F}}^H) - (z^i - \bar{z}^i) (\mathcal{F}_i^H + \bar{\mathcal{F}}_{\bar{i}}^H) - w (\Psi \mathcal{F}_\Psi^H - \bar{\Psi} \bar{\mathcal{F}}_{\bar{\Psi}}^H) . \quad (2.35)$$

Note that, as expected, the dependence on R and \bar{R} has dropped out of Σ^{Wils} . As mentioned below (2.17), F_{Wils} and $(Y^0)^2 \mathcal{F}^H(z, \Psi)$ give rise to equivalent Wilsonian Lagrangians, and thus Σ^{Wils} can only depend on \mathcal{F}^H and derivatives thereof.

Using the expressions (A.1) we find

$$\begin{aligned} \Delta_R &= 0 , \\ \Delta_\Psi &= \Sigma_\Psi^{\text{Wils}} , \\ \Delta_{\bar{\Psi}} &= 0 , \\ \Delta_{\bar{i}} - \mathcal{F}_i &= 0 , \\ \mathcal{F}_i - \bar{\Delta}_{\bar{i}} &= \Sigma_{\bar{i}}^{\text{Wils}} , \\ \mathcal{F}_{i\Psi} &= \Sigma_{i\Psi}^{\text{Wils}} , \\ \mathcal{F}_{ij} - \bar{\mathcal{F}}_{\bar{i}\bar{j}} &= \Sigma_{i\bar{j}}^{\text{Wils}} , \\ \mathcal{F}_{iR} &= 0 , \\ \Sigma_{\Psi\bar{\Psi}}^{\text{Wils}} &= 0 . \end{aligned} \quad (2.36)$$

Inserting (2.36) into (2.25) and (2.32), we obtain

$$\mathfrak{a}_\mu = \Sigma_i^{\text{Wils}} \partial_\mu z^i + \Sigma_\Psi^{\text{Wils}} \partial_\mu \Psi , \quad (2.37)$$

as well as

$$L_{\sigma}^{\text{Wils}} = |Y^0|^2 \left[i \Sigma_{A\bar{B}} \partial_{\mu} v^A \partial^{\mu} \bar{v}^B - |Y^0|^2 \Sigma_A \partial_{\mu} v^A \bar{\Sigma}_{\bar{B}} \partial^{\mu} \bar{v}^B \right], \quad (2.38)$$

where we introduced $v^A = (z^i, \Psi)$. Recalling (2.21) and defining

$$e^{-K} = i \Sigma^{\text{Wils}}, \quad (2.39)$$

we obtain

$$L_{\sigma}^{\text{Wils}} = -\partial_A \partial_{\bar{B}} K \partial_{\mu} v^A \partial^{\mu} \bar{v}^B. \quad (2.40)$$

Thus, in the Wilsonian limit, the enlarged target-space is Kähler, with the real Kähler potential K determined in terms of $\mathcal{F}^H(z, \Psi)$ via (2.35). Using (2.37) we obtain for the field strength 1-form (2.27),

$$\mathcal{F} = i \partial_A \partial_{\bar{B}} K dv^A \wedge d\bar{v}^B. \quad (2.41)$$

So far, we treated the v^A as independent fields. Now let us discuss the case when Ψ becomes a dependent field. Let us first consider the restriction to a hypersurface $\Psi = g(z)$. This is achieved by taking Υ to be the field dependent function $\Upsilon = (Y^0)^w g(z)$. Then, the sigma-model Lagrangian (2.40) and the field strength (2.41) retain their form,

$$\begin{aligned} L_{\sigma}^{\text{Wils}} &= -\partial_i \partial_{\bar{j}} K(z, \bar{z}, \Psi(z), \bar{\Psi}(\bar{z})) \partial_{\mu} z^i \partial^{\mu} \bar{z}^j, \\ \mathcal{F} &= i \partial_i \partial_{\bar{j}} K(z, \bar{z}, \Psi(z), \bar{\Psi}(\bar{z})) dz^i \wedge d\bar{z}^j. \end{aligned} \quad (2.42)$$

Next, let us consider the case when Υ is taken to be a constant parameter, in which case $\Psi = \Upsilon/(Y^0)^w$ (and similarly $\bar{\Psi}$) become functions of $(z, \bar{z}, R, \Upsilon, \bar{\Upsilon})$. Namely, using (2.21), we obtain

$$\Psi(z, \bar{z}, R, \Upsilon, \bar{\Upsilon}) = \Upsilon (i \Sigma R)^{w/2}. \quad (2.43)$$

Note that Σ depends on Ψ and $\bar{\Psi}$, so that to obtain $\Psi(z, \bar{z}, R, \Upsilon, \bar{\Upsilon})$ one has to proceed by iteration in Υ and $\bar{\Upsilon}$. Then, generically, the sigma-model will depend on the coordinates (z^i, \bar{z}^i, R) and will not retain the form (2.42). It will have additional terms of the form $\partial_{\mu} z^i \partial^{\mu} z^j$ as well as terms that involve $\partial_{\mu} R$.

2.2.2 The rigid case

Now we consider the rigid limit of (2.32). The decoupling of supergravity proceeds by setting $\mathfrak{a}_{\mu} = 0$ and setting $Y^0 = 1$, so that

$$L_{\sigma}^{\text{rigid}} = i \left[\partial_{\mu} \rho^M \partial^{\mu} \bar{z}^j \mathcal{F}_{Mj} - \text{c.c.} \right], \quad (2.44)$$

where now $\rho^M = (z^i, \bar{z}^i, \Psi = \Upsilon, \bar{\Psi} = \bar{\Upsilon})$ and $\mathcal{F} = \mathcal{F}(z, \bar{z}, \Psi, \bar{\Psi})$. This Lagrangian follows from the rigid Wilsonian Lagrangian

$$i \left[\partial_{\mu} \mathcal{F}_j^H \partial^{\mu} \bar{z}^j - \text{c.c.} \right] \quad (2.45)$$

by performing the replacement $\mathcal{F}_j^H(z, \Psi) \rightarrow \mathcal{F}_j(z, \bar{z}, \Psi, \bar{\Psi})$. When taking the fields Ψ and $\bar{\Psi}$ to be constant, i.e. when treating them as parameters, the Lagrangian (2.44) can be understood in terms of the extrinsic construction given in [7], as follows.

We consider the ambient space $V = \mathbb{C}^{2n}$ with local coordinates (z^i, w_i) , standard complex symplectic form

$$\Omega_V = dz^i \wedge dw_i, \quad (2.46)$$

and standard complex structure

$$J_V = i \left(dz^i \otimes \frac{\partial}{\partial z^i} + dw_i \otimes \frac{\partial}{\partial w_i} - d\bar{z}^i \otimes \frac{\partial}{\partial \bar{z}^i} - d\bar{w}_i \otimes \frac{\partial}{\partial \bar{w}_i} \right). \quad (2.47)$$

We define the Hermitian form γ_V (i.e. $\gamma_V(J_V X, J_V Y) = \gamma_V(X, Y) \forall X, Y \in \text{Vec}(V)$),

$$\gamma_V = i (dz^i \otimes d\bar{w}_i - dw_i \otimes d\bar{z}^i), \quad (2.48)$$

which yields the Riemannian metric

$$g_V = \text{Re } \gamma_V = i (dz^i \otimes_{\text{sym}} d\bar{w}_i - dw_i \otimes_{\text{sym}} d\bar{z}^i) \quad (2.49)$$

and the fundamental 2-form ω_V (i.e. $\omega_V(X, Y) = -g_V(J_V X, Y)$)

$$\omega_V = \text{Im } \gamma_V = (dz^i \wedge d\bar{w}_i - dw_i \wedge d\bar{z}^i), \quad d\omega_V = 0. \quad (2.50)$$

Now consider the hypersurface M in ambient space, described by $w_i = F_i(z, \bar{z})$ (we suppress the dependence on the parameters Ψ and $\bar{\Psi}$, for simplicity). Unlike the case studied in [7], this is not a holomorphic immersion. The pullback of the complex symplectic form Ω_V gives

$$\Omega_M = F_{i\bar{j}} dz^i \wedge d\bar{z}^j, \quad (2.51)$$

which is non-vanishing, and therefore $w_i = F_i(z, \bar{z})$ is not anylonger a Lagrangian immersion. The pullback of the hermitian form γ_V gives

$$\gamma_M = N_{ij} dz^i \otimes d\bar{z}^j + i \bar{F}_{i\bar{j}} dz^i \otimes dz^j - i F_{i\bar{j}} d\bar{z}^j \otimes d\bar{z}^i, \quad (2.52)$$

where N is a hermitian matrix given by

$$N_{ij} = -i (F_{ij} - \bar{F}_{i\bar{j}}). \quad (2.53)$$

The induced metric on M is

$$g_M = \text{Re } \gamma_M = N_{ij} dz^i \otimes_{\text{sym}} d\bar{z}^j + i \bar{F}_{i\bar{j}} dz^i \otimes_{\text{sym}} dz^j - i F_{i\bar{j}} d\bar{z}^j \otimes_{\text{sym}} d\bar{z}^i, \quad (2.54)$$

while the pullback of the fundamental 2-form ω_V is

$$\omega_M = \text{Im } \gamma_M = -i N_{ij} dz^i \wedge d\bar{z}^j + \bar{F}_{i\bar{j}} dz^i \wedge dz^j + F_{i\bar{j}} d\bar{z}^j \wedge d\bar{z}^i, \quad d\omega_M = 0. \quad (2.55)$$

The induced metric g_M is not hermitian with respect to the standard complex structure

$$J_M \left(\frac{\partial}{\partial z^i} \right) = i \frac{\partial}{\partial z^i}, \quad J_M \left(\frac{\partial}{\partial \bar{z}^i} \right) = -i \frac{\partial}{\partial \bar{z}^i}, \quad (2.56)$$

i.e. $g_M(J_M X, J_M Y) \neq g_M(X, Y)$ due to the terms $F_{i\bar{j}}$.

On the other hand, it is well known that given a non-degenerate 2-form ω_M and a non-degenerate metric g_M , it is possible to construct an almost complex structure J_M such that

$$\omega_M(J_M X, J_M Y) = \omega_M(X, Y). \quad (2.57)$$

The construction proceeds as follows. The non-degeneracy of both g_M and ω_M implies the existence of a linear map A such that

$$g_M(AX, Y) = -\omega_M(X, Y). \quad (2.58)$$

The adjoint A^\dagger is defined by $g(AX, Y) = g(X, A^\dagger Y)$ which, when combined with (2.58), yields $A^\dagger = -A$. The almost complex structure is then defined by

$$J_M = \left(\sqrt{AA^\dagger} \right)^{-1} A. \quad (2.59)$$

In general, however, this almost complex structure is not compatible with g_M . The compatible metric is

$$\tilde{g}_M(X, Y) = g_M \left(\sqrt{AA^\dagger} X, Y \right) = -\omega_M(X, J_M Y), \quad (2.60)$$

which does not coincide with g_M .

Let us perform the construction of J_M for the case at hand. This can be easily done by employing matrix notation. Let $F = (F_{i\bar{j}})$ and denote by F_+ and F_- its symmetric and antisymmetric parts respectively. In terms of these matrices we have

$$g_M = \begin{pmatrix} \frac{1}{2}N & -iF_+ \\ i\bar{F}_+^T & \frac{1}{2}N \end{pmatrix} \quad (2.61)$$

and

$$\omega_M = \begin{pmatrix} \frac{i}{2}N & F_- \\ -\bar{F}_-^T & -\frac{i}{2}N \end{pmatrix}. \quad (2.62)$$

Now we compute $A = \omega_M g_M^{-1}$ and obtain, by using the Schur complement,

$$A = \begin{pmatrix} iS_B \bar{S}_A^{-1} & -F^T S_A^{-1} \\ -\bar{F}^T \bar{S}_A^{-1} & -i\bar{S}_B S_A^{-1} \end{pmatrix}, \quad (2.63)$$

where

$$S_A = \frac{N}{2} - 2\bar{F}_+^T N^{-1} F_+ \xrightarrow{\text{Wil}} \frac{N}{2}, \quad (2.64)$$

$$S_B = \frac{N}{2} - 2F_- N^{-1} \bar{F}_+^T \xrightarrow{\text{Wil}} \frac{N}{2}. \quad (2.65)$$

Observe that in the Wilsonian limit ($F_{i\bar{j}} = 0$) this reduces to

$$A = i \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}, \quad (2.66)$$

and hence $AA^\dagger = I$, so that $A = J_M$, and we get a compatible triple (g_M, ω_M, J_M) , where J_M is the standard almost complex structure.

On the other hand, to first order in F , we obtain

$$A = \begin{pmatrix} i\mathbb{I} & -2F^T N^{-1} \\ -2\bar{F}^T N^{-1} & -i\mathbb{I} \end{pmatrix} \quad (2.67)$$

as well as $AA^\dagger = I$. Therefore $J_M = A$ and the compatible metric \tilde{g}_M equals g_M . Thus, to first order in F , (g_M, ω_M, J_M) forms a compatible triple.

3 Intrinsic torsion class analysis

In this section we consider the target-space metric of the deformed sigma-model (2.32) and study the intrinsic torsion of the target-space following [25]. To this end we take Υ and $\bar{\Upsilon}$ to be constant parameters. Then, $\Psi = \Upsilon/(Y^0)^w$ (and similarly $\bar{\Psi}$) become functions of $(z, \bar{z}, R, \Upsilon, \bar{\Upsilon})$, given by (2.43). Note that in (2.43), Σ depends on Ψ and $\bar{\Psi}$, so that to obtain $\Psi(z, \bar{z}, R, \Upsilon, \bar{\Upsilon})$ one has to proceed by iteration in Υ and $\bar{\Upsilon}$. For instance, to first-order in Υ , we obtain

$$\Psi^{(1)}(z, \bar{z}, R, \Upsilon) = \Upsilon \left(i\Sigma^{(0)} R \right)^{w/2}, \quad (3.1)$$

where $\Sigma^{(0)} = \Sigma(\Omega = 0)$. Proceeding in this way, the target-space metric is parametrized in terms of coordinates (z^i, \bar{z}^i, R) (with $i = 1, \dots, n$). Generically, this metric does not describe the metric of a circle fibration. We restrict ourselves to a co-dimension one hypersurface $R = \text{constant}$ and analyze the resulting geometry in terms of torsion classes. In order to use the results of [25], we focus on the case with $n = 3$ in the following, and consider a six-dimensional manifold M with Riemannian metric (we drop the symbol \otimes_{sym} in the following)

$$ds^2 = g_{i\bar{j}} dz^i d\bar{z}^j + g_{ij} dz^i dz^j + g_{i\bar{j}} d\bar{z}^i d\bar{z}^j, \quad i, j = 1, 2, 3. \quad (3.2)$$

This metric, which follows from (2.32) by setting Υ and R to a constant, depends on Ω in a rather complicated way.

We introduce complex vielbein 1-forms e_i (which are parallel with respect to the Levi-Civita connection),

$$ds^2 = \delta^{ij} \bar{e}_{\bar{i}} e_j. \quad (3.3)$$

Expressing e_i in terms of dz^i and $d\bar{z}^i$ and vice-versa,

$$\begin{aligned} e_i &= A_{ij} dz^j + B_{i\bar{j}} d\bar{z}^j \\ dz^i &= M^{ij} e_j + N^{i\bar{j}} \bar{e}_{\bar{j}}, \end{aligned} \quad (3.4)$$

we infer the relation

$$\begin{pmatrix} M & N \\ \bar{N} & \bar{M} \end{pmatrix} = \begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix}^{-1}, \quad (3.5)$$

so that

$$\begin{aligned} M &= (A - B\bar{A}^{-1}\bar{B})^{-1} , \\ N &= -A^{-1}B\bar{M} . \end{aligned} \quad (3.6)$$

The metric coefficients can be expressed in terms of A_{ij} and B_{ij} as

$$g_{i\bar{j}} = \delta_{k\bar{k}} (\bar{A}_{\bar{k}j} A_{ki} + \bar{B}_{\bar{k}i} B_{k\bar{j}}) \quad (3.7)$$

and

$$g_{ij} = \frac{\delta_{k\bar{k}}}{2} (\bar{B}_{\bar{k}i} A_{kj} + \bar{B}_{\bar{k}j} A_{ki}) . \quad (3.8)$$

Using (3.8) we express $B_{i\bar{j}}$ in terms of g_{ij} and A_{ij} as

$$B_{i\bar{j}} = \delta_{i\bar{p}} \bar{A}^{\bar{l}\bar{p}} g_{\bar{l}\bar{j}} . \quad (3.9)$$

where $\bar{A}^{\bar{l}\bar{p}} \bar{A}_{\bar{p}\bar{k}} = \delta_{\bar{p}}^{\bar{l}}$.

We introduce an almost complex structure, whose associated 2-form is (we use the same symbol J for both quantities)

$$J = -\frac{i}{2} \delta^{ij} \bar{e}_{\bar{i}} \wedge e_j \quad (3.10)$$

which, when expressed in terms of the matrices A and B , takes the form

$$J = -\frac{i}{2} (\bar{B}_{\bar{i}k} B_{i\bar{l}} - A_{ik} \bar{A}_{i\bar{l}}) dz^k \wedge d\bar{z}^l + i \bar{B}_{\bar{i}k} A_{il} dz^k \wedge dz^l + i \bar{A}_{\bar{i}k} B_{i\bar{l}} d\bar{z}^k \wedge d\bar{z}^l . \quad (3.11)$$

The triplet (M, g, J) defines a $U(3)$ structure on M . An $SU(3)$ structure is obtained by introducing a non-degenerate complex $(3, 0)$ -form ψ ,

$$\psi = e_1 \wedge e_2 \wedge e_3 . \quad (3.12)$$

Equivalently, we can define an $SU(3)$ structure on M as the triplet (M, J, ψ) , where J is a real 2-form and ψ is a complex 3-form $\psi = \psi_+ + i\psi_-$ such that the relations $J \wedge \psi_{\pm} = 0$, $\psi_+ \wedge \psi_- = \frac{2}{3} J \wedge J \wedge J \neq 0$ hold.

The failure of the holonomy group of the Levi-Civita connection of g to reduce to $SU(3)$ is measured by the so-called intrinsic torsion τ . The space to which τ belongs can be decomposed into five classes [29, 25],

$$\tau \in \mathcal{W}_1 \oplus \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5 . \quad (3.13)$$

The five torsion classes of (M, J, ψ) are defined through the decomposition of the exterior derivative of J, ψ into $SU(3)$ -modules, and they describe the failure of J, ψ to being closed, as follows [25],

$$\begin{aligned} \mathcal{W}_1 &\leftrightarrow (dJ)^{(3,0)} , & \mathcal{W}_2 &\leftrightarrow (d\psi)_0^{(2,2)} , \\ \mathcal{W}_3 &\leftrightarrow (dJ)_0^{(2,1)} , & \mathcal{W}_4 &\leftrightarrow J \wedge dJ , \\ \mathcal{W}_5 &\leftrightarrow (d\psi)^{(3,1)} . \end{aligned} \quad (3.14)$$

The subscript 0 means that only the primitive part of the form should be retained, i.e. forms that are in the kernel of $(J \wedge \bullet)$. Thus, $\beta \in \Lambda_0^{(2,2)}$ if $J \wedge \beta = 0$, and $\gamma \in \Lambda_0^{(2,1)}$ if $J \wedge \gamma = 0$. The vanishing of both \mathcal{W}_1 and \mathcal{W}_2 implies that the manifold is hermitian, while the vanishing of the four torsion classes $\mathcal{W}_1, \dots, \mathcal{W}_4$ implies that the manifold is Kähler.

We proceed to determine the torsion classes. To do so, we write de_i as

$$de_i = (\mathcal{O}_1)_i^{mn} e_m \wedge e_n + (\mathcal{O}_2)_i^{\bar{m}\bar{n}} \bar{e}_{\bar{m}} \wedge e_n + (\mathcal{O}_3)_i^{\bar{m}\bar{n}} \bar{e}_{\bar{m}} \wedge \bar{e}_{\bar{n}}. \quad (3.15)$$

The torsion classes are then expressed as

$$\begin{aligned} \mathcal{W}_1 &\leftrightarrow \bar{\mathcal{Z}}^{jk\bar{j}\bar{k}} \psi, \\ \tilde{\mathcal{W}}_2 &\leftrightarrow \mathcal{Z}^{\bar{m}\bar{n}jk} \bar{e}_{\bar{m}\bar{n}} \wedge e_{jk}, \\ \tilde{\mathcal{W}}_3 &\leftrightarrow \mathcal{C}^{ij\bar{k}} e_{ij} \wedge \bar{e}_{\bar{k}}, \\ \mathcal{W}_4 &\leftrightarrow \mathcal{R}^{\bar{n}\bar{k}} \psi \wedge \bar{e}_{\bar{n}\bar{k}} + \mathcal{S}^{nk} \bar{\psi} \wedge e_{nk}, \\ \mathcal{W}_5 &\leftrightarrow (\mathcal{O}_2)_i^{\bar{m}\bar{i}} \bar{e}_{\bar{m}} \wedge \psi, \end{aligned} \quad (3.16)$$

where $e_{jk} = e_j \wedge e_k$, ϵ_{ijk} denotes the totally anti-symmetric epsilon tensor, and where we introduced the following tensors,

$$\begin{aligned} \mathcal{C}^{ij\bar{k}} &= -\frac{1}{2} \left[\left(\delta^{\bar{l}j} (\bar{\mathcal{O}}_2)_{\bar{l}}^{i\bar{k}} + \delta^{\bar{k}l} (\mathcal{O}_1)_l^{ij} \right) - (i \leftrightarrow j) \right], \\ \mathcal{E}^{ij\bar{k}} = -\bar{\mathcal{C}}^{\bar{j}\bar{k}i} &= \frac{1}{2} \left[\left(\delta^{\bar{l}i} (\bar{\mathcal{O}}_1)_{\bar{l}}^{\bar{j}\bar{k}} - \delta^{\bar{j}l} (\mathcal{O}_2)_l^{\bar{k}i} \right) - (\bar{j} \leftrightarrow \bar{k}) \right], \\ \mathcal{R}^{\bar{n}\bar{k}} &= -\frac{1}{2} \left[\delta^{\bar{n}l} \mathcal{C}^{ij\bar{k}} \epsilon_{lij} - (\bar{n} \leftrightarrow \bar{k}) \right], \\ \mathcal{S}^{nk} = \overline{(\mathcal{R}^{\bar{n}\bar{k}})} &= \frac{1}{2} \left[\delta^{n\bar{m}} \mathcal{E}^{k\bar{j}\bar{l}} \epsilon_{\bar{m}\bar{j}\bar{l}} - (n \leftrightarrow k) \right], \\ \mathcal{Z}^{\bar{m}\bar{n}jk} &= (\mathcal{O}_3)_i^{\bar{m}\bar{n}} \epsilon^{ijk}. \end{aligned} \quad (3.17)$$

Note that in (3.16) we used $\tilde{\mathcal{W}}$ on the left hand side of some of the equations, to indicate that in order to obtain the primitive part of the forms, we have to impose the following conditions. The torsion class \mathcal{W}_2 has to satisfy $J \wedge \mathcal{W}_2 = 0$. We compute

$$J \wedge \tilde{\mathcal{W}}_2 \propto \left(\mathcal{Z}^{\bar{j}\bar{k}jk} - \mathcal{Z}^{\bar{k}\bar{j}jk} \right) \bar{\psi} \wedge \psi = 2\mathcal{Z}^{\bar{j}\bar{k}jk} \bar{\psi} \wedge \psi, \quad (3.18)$$

which is proportional to the torsion class \mathcal{W}_1 . Thus, we conclude that when \mathcal{W}_1 vanishes, $(d\psi)^{(2,2)}$ is primitive, i.e. $(d\psi)^{(2,2)} = (d\psi)_0^{(2,2)}$, and $\tilde{\mathcal{W}}_2 = \mathcal{W}_2$.

The torsion class \mathcal{W}_3 , on the other hand, has to satisfy $J \wedge \mathcal{W}_3 = 0$. We compute

$$J \wedge \tilde{\mathcal{W}}_3 = \mathcal{C}^{ij\bar{k}} \epsilon_{ijl} \delta^{lp} \psi \wedge \bar{e}_{\bar{k}\bar{p}}, \quad (3.19)$$

which is proportional to the torsion class \mathcal{W}_4 . Thus, when \mathcal{W}_4 vanishes, we have $\tilde{\mathcal{W}}_3 = \mathcal{W}_3$.

The coefficients (\mathcal{O}) appearing in (3.17) can be expressed in terms of the matrices A, B, M

and N introduced in (3.4). We obtain

$$\begin{aligned}
(\mathcal{O}_1)_i^{mn} &= \frac{1}{2} \left(\left(\partial_k A_{ij} M^{km} + \partial_{\bar{k}} A_{ij} \bar{N}^{\bar{k}m} \right) M^{jn} - (m \leftrightarrow n) \right) \\
&\quad + \frac{1}{2} \left(\left(\partial_k B_{i\bar{j}} M^{km} + \partial_{\bar{k}} B_{i\bar{j}} \bar{N}^{\bar{k}m} \right) \bar{N}^{\bar{j}n} - (m \leftrightarrow n) \right) , \\
(\mathcal{O}_2)_i^{\bar{m}n} &= \partial_k A_{ij} \left(M^{jn} N^{k\bar{m}} - M^{kn} N^{j\bar{m}} \right) + \partial_{\bar{k}} A_{ij} \left(\bar{M}^{\bar{k}\bar{m}} M^{jn} - \bar{N}^{\bar{k}n} N^{j\bar{m}} \right) \\
&\quad + \partial_k B_{i\bar{j}} \left(N^{k\bar{m}} \bar{N}^{\bar{j}n} - M^{kn} \bar{M}^{\bar{j}\bar{m}} \right) + \partial_{\bar{k}} B_{i\bar{j}} \left(\bar{M}^{k\bar{m}} \bar{N}^{\bar{j}n} - \bar{N}^{\bar{k}n} \bar{M}^{\bar{j}\bar{m}} \right) , \\
(\mathcal{O}_3)_i^{\bar{m}\bar{n}} &= \frac{1}{2} \left(\left(\partial_k A_{ij} N^{k\bar{m}} + \partial_{\bar{k}} A_{ij} \bar{M}^{\bar{k}\bar{m}} \right) N^{j\bar{n}} - (\bar{m} \leftrightarrow \bar{n}) \right) \\
&\quad + \frac{1}{2} \left(\left(\partial_k B_{i\bar{j}} N^{k\bar{m}} + \partial_{\bar{k}} B_{i\bar{j}} \bar{M}^{\bar{k}\bar{m}} \right) \bar{M}^{\bar{j}\bar{n}} - (\bar{m} \leftrightarrow \bar{n}) \right) . \tag{3.20}
\end{aligned}$$

There are two cases to consider, namely $B_{i\bar{j}} = 0$ and $B_{i\bar{j}} \neq 0$. They can be understood as follows. The metric (3.2), which is obtained from (2.32), depends on Ω in a rather complicated way. We may expand the metric coefficients in powers of Ω and compute the torsion classes order by order in Ω . When $\Omega = 0$, we have $g_{ij} = 0$ and we infer from (3.9) that $B_{i\bar{j}} = 0$. In this case the metric is Kähler, since it corresponds to the sigma-model metric in the absence of a chiral background. On the other hand, when $\Omega \neq 0$ we have $g_{ij} \neq 0$, as discussed at the end of subsection 2.2.1, and hence also $B_{i\bar{j}} \neq 0$.

We first consider the case when $B_{i\bar{j}} = 0$ and verify that the vanishing of the first four torsion classes implies that the manifold is Kähler. We infer from (3.6) that $N^{i\bar{j}} = 0$, and from (3.8) and (3.7) we obtain $g_{ij} = 0$ and $g_{i\bar{j}} = \delta_{k\bar{k}} A_{ki} \bar{A}_{\bar{k}\bar{j}}$. It follows that $(\mathcal{O}_3) = 0$, so that $\mathcal{W}_1 = \mathcal{W}_2 = 0$, and hence the manifold is hermitian. For (\mathcal{O}_1) and (\mathcal{O}_2) we obtain

$$(\mathcal{O}_1)_i^{mn} = \frac{1}{2} \left(\partial_k A_{ij} M^{km} M^{jn} - (m \leftrightarrow n) \right) \tag{3.21}$$

and

$$(\mathcal{O}_2)_i^{\bar{m}n} = \partial_{\bar{k}} A_{ij} \bar{M}^{\bar{k}\bar{m}} M^{jn} . \tag{3.22}$$

This results in

$$\mathcal{C}^{ij\bar{k}} = -\frac{1}{2} \left[\partial_n g_{l\bar{m}} M^{lj} M^{ni} \bar{M}^{\bar{k}\bar{m}} - (i \leftrightarrow j) \right] , \tag{3.23}$$

which vanishes iff $\partial_n g_{l\bar{m}} = \partial_l g_{n\bar{m}}$, in which case

$$\mathcal{W}_3 = \mathcal{W}_4 = 0. \tag{3.24}$$

Thus we recover the well known fact that the vanishing of the first four torsion classes implies that the manifold is Kähler.

We now briefly discuss the torsion class \mathcal{W}_5 when $B_{i\bar{j}} = 0$. In this case \mathcal{W}_5 is proportional to

$$(\mathcal{O}_2)_i^{\bar{m}i} = \partial_{\bar{k}} A_{ij} \bar{M}^{\bar{k}\bar{m}} M^{ji} . \tag{3.25}$$

Taking M^{ij} to be diagonal,

$$M^{ij} = \lambda_i(z, \bar{z}) \delta^{ij} \quad , \quad \lambda_i \in \mathbb{C} , \tag{3.26}$$

we have $\ln \sqrt{\det g_{i\bar{j}}} = -\frac{1}{2} \ln \prod_{i=1}^3 (\lambda_i \bar{\lambda}_i)$, and we obtain (no summation over \bar{m} here)

$$(\mathcal{O}_2)_i^{\bar{m}i} = -\sum_{i=1}^3 \frac{\bar{\lambda}_{\bar{m}}}{\lambda_i} \partial_{\bar{m}} \lambda_i = -\bar{\lambda}_{\bar{m}} \partial_{\bar{m}} \ln \prod_{i=1}^3 \lambda_i. \quad (3.27)$$

In order for \mathcal{W}_5 to vanish we must have

$$\partial_{\bar{m}} \ln \prod_{i=1}^3 \lambda_i = 0 \quad , \quad m = 1, 2, 3, \quad (3.28)$$

which, together with the complex-conjugate equation, implies the vanishing the first Chern class

$$c_1 = \left[\frac{1}{2\pi} \mathcal{R} \right], \quad (3.29)$$

where \mathcal{R} is the Ricci form

$$\mathcal{R} = i\partial\bar{\partial} \ln \sqrt{\det g_{i\bar{j}}}. \quad (3.30)$$

Next, we consider the case when $B_{i\bar{j}} \neq 0$, in which case $g_{ij} \neq 0$. We may expand the metric coefficients in powers of Ω . In the following, we compute the four torsion classes $\mathcal{W}_1, \dots, \mathcal{W}_4$ to first order in Ω . At first order in Ω we have $g_{ij} \neq 0$ and hence, according to (3.9), this means that we will work at first order in $B_{i\bar{j}}$. We therefore expand the metric (3.2) in powers of B as

$$ds^2 = \left(g_{i\bar{j}}^{(0)} + g_{i\bar{j}}^{(1)} \right) dz^i d\bar{z}^j + g_{ij}^{(1)} dz^i dz^j + g_{i\bar{j}}^{(1)} d\bar{z}^i d\bar{z}^j, \quad (3.31)$$

where the superscript indicates the power of B (or, equivalently, the power of Ω). For later use, we introduce the Christoffel symbol associated to the Kähler metric $g^{(0)}$,

$$\Gamma_{\bar{k}\bar{l}}^{\bar{p}} = g^{(0)\bar{p}r} \partial_{\bar{k}} g_{r\bar{l}}^{(0)}. \quad (3.32)$$

We now compute the coefficients (\mathcal{O}) given in (3.20) to first order in $B_{i\bar{j}}$. Observe that $N^{i\bar{j}}$ is then of first order in B . First, we rewrite (\mathcal{O}_3) given in (3.20) as

$$(\mathcal{O}_3)_i^{\bar{m}\bar{n}} = \frac{1}{2} \left[A_{il} \bar{M}^{\bar{j}\bar{n}} \left(N^{k\bar{m}} \partial_k \left(A^{ls} B_{s\bar{j}} \right) + \bar{M}^{\bar{k}\bar{m}} \partial_{\bar{k}} \left(A^{ls} B_{s\bar{j}} \right) \right) - (\bar{m} \leftrightarrow \bar{n}) \right]. \quad (3.33)$$

To first order in $B_{i\bar{j}}$ this yields,

$$(\mathcal{O}_3)_i^{\bar{m}\bar{n}} = \frac{A_{il}^{(0)}}{2} \left(\bar{M}_0^{\bar{j}\bar{n}} \bar{M}_0^{\bar{k}\bar{m}} \partial_{\bar{k}} \left(g^{(0)\bar{p}l} g_{\bar{p}\bar{j}}^{(1)} \right) - (\bar{m} \leftrightarrow \bar{n}) \right), \quad (3.34)$$

where we used (3.9), and where $g^{(0)\bar{p}l} = \bar{A}^{\bar{p}\bar{k}} A^{lk}$. Using (3.32) we have

$$(\mathcal{O}_3)_i^{\bar{m}\bar{n}} = \frac{\delta_{i\bar{l}}}{2} \bar{M}_0^{\bar{j}\bar{n}} \bar{M}_0^{\bar{k}\bar{m}} \bar{M}_0^{\bar{l}\bar{i}} \left(\partial_{\bar{k}} g_{\bar{l}\bar{j}}^{(1)} - \Gamma_{\bar{k}\bar{l}}^{\bar{p}} g_{\bar{p}\bar{j}}^{(1)} - (\bar{j} \leftrightarrow \bar{k}) \right). \quad (3.35)$$

Introducing the covariant derivative with respect to the Levi-Civita connection (3.32),

$$\mathcal{D}_{\bar{k}} g_{\bar{l}\bar{j}}^{(1)} = \partial_{\bar{k}} g_{\bar{l}\bar{j}}^{(1)} - \Gamma_{\bar{k}\bar{l}}^{\bar{p}} g_{\bar{p}\bar{j}}^{(1)} - \Gamma_{\bar{k}\bar{j}}^{\bar{p}} g_{\bar{l}\bar{p}}^{(1)}, \quad (3.36)$$

we obtain

$$(\mathcal{O}_3)_i^{\bar{m}\bar{n}} = \frac{\delta_{i\bar{i}}}{2} \bar{M}_0^{\bar{m}} \bar{M}_0^{\bar{k}\bar{m}} \bar{M}_0^{\bar{l}\bar{i}} \left(\mathcal{D}_{\bar{k}} g_{\bar{l}\bar{j}}^{(1)} - \mathcal{D}_{\bar{j}} g_{\bar{l}\bar{k}}^{(1)} \right). \quad (3.37)$$

It follows that to first order in Ω ,

$$\bar{\mathcal{W}}_1 \propto (\mathcal{O}_3)_i^{\bar{m}\bar{n}} \epsilon^{imn} = 0, \quad (3.38)$$

and hence $\tilde{\mathcal{W}}_2 = \mathcal{W}_2$.

Similarly, we compute

$$\mathcal{C}^{ij\bar{k}} = -\frac{1}{2} \bar{M}_0^{\bar{m}\bar{k}} M_0^{li} M_0^{nj} \left(\mathcal{D}_l g_{n\bar{m}}^{(1)} - \mathcal{D}_n g_{l\bar{m}}^{(1)} \right). \quad (3.39)$$

This determines $\tilde{\mathcal{W}}_3$ and \mathcal{W}_4 to first order in Ω . Notice that (3.39) vanishes provided that $\partial_l g_{n\bar{m}}^{(1)} = \partial_n g_{l\bar{m}}^{(1)}$, i.e. if $g_{n\bar{m}}^{(1)}$ is Kähler.

Summarizing, at first order in Ω we find that $\mathcal{W}_1 = 0$, while \mathcal{W}_2 is given by (3.37). A non-vanishing class \mathcal{W}_2 implies that the almost complex structure is not integrable. The torsion classes $\tilde{\mathcal{W}}_3$ and \mathcal{W}_4 are determined by (3.39).

In the next section, we give an example of a deformed sigma-model metric that, to first order in Ω , has $\mathcal{W}_1 = \mathcal{W}_3 = \mathcal{W}_4 = 0$. Thus, the target manifold is almost-Kähler, since $dJ = 0$ [29].

4 Example

In the following we identify the chiral background with the square of the Weyl multiplet, so that the model we will examine in this section describes the coupling to R^2 -terms. Thus the Weyl weight w equals $w = 2$.

The model we consider is based on a function $F(Y, \bar{Y}, \Upsilon, \bar{\Upsilon})$ that is homogeneous of degree two and specified by (see (1.2))

$$\begin{aligned} F^{(0)}(Y) &= -(Y^0)^2 z^1 z^2 z^3, \\ \Omega(Y, \bar{Y}, \Upsilon, \bar{\Upsilon}) &= \Upsilon g(z) + \bar{\Upsilon} \bar{g}(\bar{z}) + (\Upsilon + \bar{\Upsilon}) h(z, \bar{z}), \end{aligned} \quad (4.1)$$

where h is a real function, while g is holomorphic, and

$$S = -iz^1 = -i \frac{Y^1}{Y^0}, \quad T = -iz^2 = -i \frac{Y^2}{Y^0}, \quad U = -iz^3 = -i \frac{Y^3}{Y^0}. \quad (4.2)$$

The harmonic part of Ω determines the Wilsonian Lagrangian. We take $g(z)$ and $h(z, \bar{z})$ to be decomposable as

$$g(z) = \sum_{i=1}^3 f^{(i)}(z^i), \quad h(z, \bar{z}) = \sum_{i=1}^3 p^{(i)}(z^i, \bar{z}^i), \quad (4.3)$$

with real $p^{(i)}$. Then, the model describes an $N = 2$ STU-type model [30] in the presence of R^2 -interactions [31]. Ω may also receive corrections that are of higher order in Υ and $\bar{\Upsilon}$.

Here, we will work to first order in Υ and $\bar{\Upsilon}$. We may therefore identify first-order in Υ with first-order in Ω , which is the order employed in the torsion analysis of the previous section.

Inserting (4.1) into (2.32) results in a rather complicated target-space metric which, to first order in Υ , contains $d\Psi, d\bar{\Psi}$ and dR -legs. To simplify the metric, we perform a coordinate transformation whose form can be motivated by duality symmetry considerations, as follows.

Suppose we demand the model to have S- and T-duality symmetries. In the presence of a chiral background, the S- and T-duality transformation rules get modified and involve derivatives of Ω . For instance, under S-duality, the fields Y^0, S, T, U transform as follows [22],

$$\begin{aligned} S &\rightarrow \frac{aS - ib}{icS + d} \quad , \quad Y^0 \rightarrow \Delta_S Y^0 \quad , \quad \Delta_S = icS + d \quad , \\ T &\rightarrow T + \frac{2ic}{\Delta_S(Y^0)^2} \frac{\partial \Omega}{\partial U} \quad , \quad U \rightarrow U + \frac{2ic}{\Delta_S(Y^0)^2} \frac{\partial \Omega}{\partial T} \quad , \end{aligned} \quad (4.4)$$

where a, b, c, d are integer-valued parameters that satisfy $ad - bc = 1$ and parametrize (a subgroup of) $\text{SL}(2, \mathbb{Z})$. Here we recall that Υ does not transform, since it is invariant under symplectic transformations. We obtain similar transformation rules under T- and U-duality if we assume that the model has triality symmetry. These transformation rules then follow from (4.4) by appropriately interchanging S, T and U . Furthermore, for the model to possess S- and T-dualities, we need to take Υ to be real. Then, at first order in Υ , the model based on (4.1) is invariant under S-duality transformations (4.4) provided that the derivatives of Ω with respect to S, T, U transform as [22]

$$\left(\frac{\partial \Omega}{\partial S} \right)' = \Delta_S^2 \frac{\partial \Omega}{\partial S} \quad , \quad \left(\frac{\partial \Omega}{\partial T} \right)' = \frac{\partial \Omega}{\partial T} \quad , \quad \left(\frac{\partial \Omega}{\partial U} \right)' = \frac{\partial \Omega}{\partial U} \quad . \quad (4.5)$$

At higher order in Υ , these expressions get corrected according to [22]. Similar considerations hold for T-duality transformations. In general, S- and T-duality invariance can only be achieved if the real functions $p^{(i)}$ in (4.3) are chosen in a suitable way.

Now we redefine the field S in such a way that the new field \tilde{S} transforms in the usual way under S-duality [10],

$$\tilde{S} \rightarrow \frac{a\tilde{S} - ib}{ic\tilde{S} + d} \quad , \quad (4.6)$$

and is invariant under T-dualities. To first order in Υ , this is achieved by the following combination,¹

$$\tilde{S} = S + \frac{2}{(Y^0)^2} \left(\frac{\partial_T \Omega}{U + \bar{U}} + \frac{\partial_U \Omega}{T + \bar{T}} \right) \quad . \quad (4.8)$$

Using (4.4) and (4.5) it can be readily checked that (4.8) transforms according to (4.6). Similarly, it can be verified that \tilde{S} is invariant under T- and U-duality transformations. At first order in Υ , the transformation laws for T- and U-duality are obtained from (4.4) by interchanging S with T and U , respectively.

¹We note that the combination introduced in [10] differs from (4.8) by a term that is invariant under T-dualities,

$$- 2 \frac{(S + \bar{S})}{|Y^0|^2} \frac{\partial_{\bar{S}} \Omega}{(T + \bar{T})(U + \bar{U})} \quad . \quad (4.7)$$

Under S-duality, this term transforms in the same way as the difference $\tilde{S} - S$ in (4.8).

We also introduce new fields \tilde{T} and \tilde{U} that are defined as in (4.8) by interchanging S with T and U , respectively.

Having motivated the coordinate transformation (4.8) by using duality symmetry considerations, we apply it to the model (4.1) and obtain

$$\tilde{z}^1 = z^1 - 2i\Psi \left[\frac{g_2 + h_2}{z^3 - \bar{z}^3} + \frac{g_3 + h_3}{z^2 - \bar{z}^2} \right] - 2iR^2 \bar{\Psi} \left[\frac{h_2}{z^3 - \bar{z}^3} + \frac{h_3}{z^2 - \bar{z}^2} \right], \quad (4.9)$$

where $g_2 = \partial g / \partial z^2$, $h_2 = \partial h / \partial z^2$, etc. Similar expressions hold for \tilde{z}^2 and \tilde{z}^3 . Then, expressing the target-space metric (2.32) in terms of the new coordinates \tilde{z}^i , we obtain that to first order in Υ , the target-space metric takes the form (3.31), with no extra $d\Psi$, $d\bar{\Psi}$ and dR -legs. The metric (3.31) is now expressed in terms of the new coordinates. Its metric coefficients depend on \tilde{z}^i , $\bar{\tilde{z}}^i$, Ψ , $\bar{\Psi}$ and R . Using (4.3) we find

$$g_{i\bar{j}}^{(0)} = \frac{\delta_{i\bar{j}}}{2(\tilde{z}^i - \bar{\tilde{z}}^i)^2}, \quad g_{ij}^{(1)} = 0, \quad (4.10)$$

while the metric components $g_{ij}^{(1)}$ are off-diagonal and take the form

$$g_{12}^{(1)} = -\frac{2i}{(\bar{Y}^0)^2} \frac{\partial_3 \Omega}{(\tilde{z}^1 - \bar{\tilde{z}}^1)^2 (\tilde{z}^2 - \bar{\tilde{z}}^2)^2} = -2i \frac{[\bar{\Psi}(\bar{g}_3 + h_3) + \bar{R}^2 \Psi h_3]}{(\tilde{z}^1 - \bar{\tilde{z}}^1)^2 (\tilde{z}^2 - \bar{\tilde{z}}^2)^2}, \quad (4.11)$$

and similarly for the components $g_{13}^{(1)}$ and $g_{23}^{(1)}$, which follow from (4.11) by suitably interchanging \tilde{z}^1 , \tilde{z}^2 and \tilde{z}^3 . Observe that when the model has S- and T-duality symmetries, the combination (4.11) is such that $g_{ij}^{(1)} d\tilde{z}^i d\tilde{z}^j$ is duality invariant at first order in Υ .

Now we return to the torsion analysis. Using the above target-space metric, we immediately infer that (3.39) vanishes, so that $\mathcal{W}_3 = \mathcal{W}_4 = 0$. On the other hand, using (3.16), (3.17) and (3.37) we find that the torsion 4-form $\tilde{\mathcal{W}}_2$ is proportional to

$$\tilde{\mathcal{W}}_2 \propto \bar{e}_{\bar{m}\bar{n}} \wedge \left(e_{23} \mathcal{D}_{\bar{m}} g_{1\bar{n}}^{(1)} + e_{31} \mathcal{D}_{\bar{m}} g_{2\bar{n}}^{(1)} + e_{12} \mathcal{D}_{\bar{m}} g_{3\bar{n}}^{(1)} \right). \quad (4.12)$$

Focussing on the terms proportional to e_{23} we obtain

$$e_{23} \wedge \left(\bar{e}_{1\bar{2}} \mathcal{D}_{\bar{1}} g_{1\bar{2}}^{(1)} + \bar{e}_{1\bar{3}} \mathcal{D}_{\bar{1}} g_{1\bar{3}}^{(1)} + \bar{e}_{2\bar{3}} \left(\mathcal{D}_2 g_{1\bar{3}}^{(1)} - \mathcal{D}_3 g_{1\bar{2}}^{(1)} \right) \right), \quad (4.13)$$

where we used that the metric coefficients $g_{ij}^{(1)}$ are off-diagonal. Similar expressions hold for the other terms in (4.12). Using these, we verify that $\tilde{\mathcal{W}}_2 \wedge J = 0$, which implies that $\tilde{\mathcal{W}}_2 = \mathcal{W}_2$, as was expected since $\mathcal{W}_1 = 0$ (see (3.38)). Evaluating (4.13) using (4.3) we obtain

$$\begin{aligned} & 2i e_{23} \wedge \frac{(\Psi + R^2 \bar{\Psi}) \bar{e}_{3\bar{2}} h_{33} + \bar{e}_{1\bar{2}} \partial_1 (\Psi + R^2 \bar{\Psi}) h_3 + \bar{e}_{3\bar{2}} \partial_3 (\Psi + R^2 \bar{\Psi}) h_3}{(\tilde{z}^1 - \bar{\tilde{z}}^1)^2 (\tilde{z}^2 - \bar{\tilde{z}}^2)^2} \\ & + 2i e_{23} \wedge \frac{(\Psi + R^2 \bar{\Psi}) \bar{e}_{2\bar{3}} h_{22} + \bar{e}_{1\bar{3}} \partial_1 (\Psi + R^2 \bar{\Psi}) h_2 + \bar{e}_{2\bar{3}} \partial_2 (\Psi + R^2 \bar{\Psi}) h_2}{(\tilde{z}^1 - \bar{\tilde{z}}^1)^2 (\tilde{z}^3 - \bar{\tilde{z}}^3)^2}. \end{aligned} \quad (4.14)$$

This vanishes in the Wilsonian limit provided we take Ψ to be an independent field, in agreement with the discussion presented in subsection 2.2.1. On the other hand, when turning on non-holomorphic terms (which are encoded in $h(z, \bar{z})$), the combination (4.14) is non-vanishing. Similar considerations hold for the other terms in (4.12). Thus we conclude that the torsion class \mathcal{W}_2 becomes non-vanishing in the presence of non-holomorphic terms and/or when taking Ψ to be a dependent field, as in (3.1). Finally, we note that also the torsion class \mathcal{W}_5 is non-vanishing for this model, since (3.27) is already non-vanishing when $\Omega = 0$.

5 Real coordinates and the Hesse potential

In [24] a theorem was presented according to which an arbitrary point-particle Lagrangian $\mathcal{L}(\phi, \dot{\phi})$ depending on coordinates ϕ^i and velocities $\dot{\phi}$ can be formulated in terms of a complex function $F(x, \bar{x})$, where $x^i = \frac{1}{2}(\phi^i + i\dot{\phi})$, such that the canonical variables $(\phi^i, \pi_i = \partial\mathcal{L}/\partial\dot{\phi}^i)$ equal

$$\begin{aligned}\phi^i &= 2 \operatorname{Re} x^i, \\ \pi_i &= 2 \operatorname{Re} F_i, \quad F_i = \frac{\partial F(x, \bar{x})}{\partial x^i}.\end{aligned}\tag{5.1}$$

The function $F(x, \bar{x})$ takes the form displayed in (1.2), and the Lagrangian \mathcal{L} and the Hamiltonian H are expressed in terms of F as

$$\begin{aligned}\mathcal{L} &= 4(\operatorname{Im} F - \Omega), \\ H &= \dot{\phi}^i \pi_i - \mathcal{L}(\phi, \dot{\phi}) = -i(x^i \bar{F}_i - \bar{x}^i F_i) - 4 \operatorname{Im} \left(F^{(0)} - \frac{1}{2} x^i F_i^{(0)} \right) - 2(2\Omega - x^i \Omega_i - \bar{x}^i \Omega_{\bar{i}}),\end{aligned}\tag{5.2}$$

where $F_i^{(0)} = \partial F^{(0)} / \partial x^i$, $\Omega_i = \partial \Omega / \partial x^i$, and similarly for their complex conjugates.

In the absence of a chiral background, it is known that the Hesse potential underlying the real formulation of special geometry [4, 5, 32, 9, 12] equals the Hamiltonian H displayed above (which in this case reads $H = \dot{\phi}^i \pi_i - 4 \operatorname{Im} F^{(0)}$). This continues to hold in the presence of a chiral background, since H given in (5.2) incorporates the chiral background in a manner that respects the symplectic nature of H . We proceed to verify this explicitly. We follow the exposition of [12].

We consider the function $F(Y, \bar{Y}, \Upsilon, \bar{\Upsilon})$ given in (1.2), which is homogeneous of degree two. We follow the theorem outlined above and introduce real coordinates² (x^I, y_I)

$$Y^I = x^I + i u^I(x, y), \quad F_I = y_I + i v_I(x, y).\tag{5.3}$$

as well as

$$\begin{aligned}\mathcal{L}(Y, \bar{Y}, \Upsilon, \bar{\Upsilon}) &= \operatorname{Im} F - \Omega, \\ H(x, y, \Upsilon, \bar{\Upsilon}) &= u^I y_I - \mathcal{L}.\end{aligned}\tag{5.4}$$

Here, we have scaled out a factor 4 relative to (5.2).

We introduce the symplectic vector

$$V_a = \begin{pmatrix} Y^I \\ F_I(Y, \bar{Y}, \Upsilon, \bar{\Upsilon}) \end{pmatrix},\tag{5.5}$$

and decompose it into real and imaginary parts,

$$V_a = \mathcal{R}_a + i \mathcal{I}_a = \begin{pmatrix} x^I \\ y_I \end{pmatrix} + i \begin{pmatrix} u^I \\ v_I \end{pmatrix},\tag{5.6}$$

²The x^I introduced here should not be confused with the x^i in (5.1), which are complex.

where

$$\begin{pmatrix} u^I \\ v_I \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial y_I} \\ -\frac{\partial H}{\partial x^I} \end{pmatrix}. \quad (5.7)$$

We rewrite (5.7) as

$$\mathcal{I}_a = \omega_{ab} H_b, \quad (5.8)$$

where ω_{ab} denotes the symplectic matrix

$$\omega = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}, \quad (5.9)$$

and H_a denotes

$$H_a = \begin{pmatrix} \frac{\partial H}{\partial x^I} \\ \frac{\partial H}{\partial y_I} \end{pmatrix}. \quad (5.10)$$

Using these, we obtain for (2.15),

$$\mathcal{A}_\mu = (\omega \mathcal{R})_a \partial_\mu \mathcal{R}_a + (\omega H)_a \partial_\mu H_a, \quad (5.11)$$

and for the sigma-model Lagrangian (2.14),

$$L_\sigma = 2\partial^\mu \mathcal{R}_a \partial_\mu H_a - [(\omega \mathcal{R})_a \partial_\mu \mathcal{R}_a + (\omega H)_a \partial_\mu H_a]^2. \quad (5.12)$$

Here H_a depends on the real coordinates \mathcal{R} and on the chiral background field Υ and $\bar{\Upsilon}$.

Now we consider the case when the chiral background field Υ is taken to be constant. Then we have the relation

$$\partial_\mu H_a = H_{ab} \partial_\mu \mathcal{R}_b, \quad (5.13)$$

where H_{ab} denotes the Hessian matrix

$$H_{ab} = \begin{pmatrix} \frac{\partial^2 H}{\partial x^I \partial x^J} & \frac{\partial^2 H}{\partial y_I \partial x^J} \\ \frac{\partial^2 H}{\partial x^I \partial y_J} & \frac{\partial^2 H}{\partial y_I \partial y_J} \end{pmatrix}. \quad (5.14)$$

Inserting (5.13) into (5.12) yields

$$L_\sigma = \left[2H_{ab} - \left((\omega \mathcal{R})_a + (\omega H)_c H_{ca} \right) \left((\omega \mathcal{R})_b + (\omega H)_d H_{db} \right) \right] \partial^\mu \mathcal{R}_a \partial_\mu \mathcal{R}_b. \quad (5.15)$$

In the rigid case, only the first term is present, which is encoded in the Hessian H_{ab} . Thus, H given in (5.4) serves as the Hesse potential for real special geometry in the presence of a chiral background.

Next, we compute the Hessian matrix H_{ab} . Using (5.7) we express the Hessian as

$$H_{ab} = \begin{pmatrix} -\frac{\partial v_I}{\partial x^I} \big|_y & -\frac{\partial v_I}{\partial y_I} \big|_x \\ \frac{\partial u^J}{\partial x^I} \big|_y & \frac{\partial u^J}{\partial y_I} \big|_x \end{pmatrix}, \quad (5.16)$$

and infer

$$\begin{aligned}\frac{\partial u^J}{\partial x^I}\big|_y &= -\frac{\partial v_I}{\partial y_J}\big|_x, \\ \frac{\partial v_J}{\partial x^I}\big|_y &= \frac{\partial v_I}{\partial x^J}\big|_y, \\ \frac{\partial u^J}{\partial y_I}\big|_x &= \frac{\partial u^I}{\partial y_J}\big|_x.\end{aligned}\tag{5.17}$$

We proceed to compute these quantities. To this end, we introduce the combinations

$$\begin{aligned}R_{IJ} &= F_{IJ} + \bar{F}_{\bar{I}\bar{J}}, \\ N_{IJ} &= -i(F_{IJ} - \bar{F}_{\bar{I}\bar{J}}),\end{aligned}\tag{5.18}$$

as well as

$$\begin{aligned}[R_{\pm}]_{IJ} &= R_{IJ} \pm 2\operatorname{Re} F_{I\bar{J}}, \\ [N_{\pm}]_{IJ} &= N_{IJ} \pm 2\operatorname{Im} F_{I\bar{J}}.\end{aligned}\tag{5.19}$$

Observe that

$$\begin{aligned}[R_{\pm}]^T &= R_{\mp}, \\ [N_{\pm}]^T &= N_{\pm}.\end{aligned}\tag{5.20}$$

We compute the Jacobian J associated with the change of variables $(Y, \bar{Y}) \rightarrow (x, y)$,

$$J = \frac{\partial(x, y)}{\partial(Y, \bar{Y})} = \frac{1}{2} \begin{pmatrix} \delta_J^I & \delta_J^I \\ F_{IJ} + \bar{F}_{\bar{I}\bar{J}} & F_{I\bar{J}} + \bar{F}_{\bar{I}\bar{J}} \end{pmatrix}.\tag{5.21}$$

The inverse Jacobian is given by

$$J^{-1} = \frac{\partial(Y, \bar{Y})}{\partial(x, y)} = \begin{pmatrix} \delta^J_K + i\frac{\partial u^J}{\partial x^K}\big|_y & i\frac{\partial u^J}{\partial y_K}\big|_x \\ \delta^J_K - i\frac{\partial u^J}{\partial x^K}\big|_y & -i\frac{\partial u^J}{\partial y_K}\big|_x \end{pmatrix}.\tag{5.22}$$

Then, by imposing $J^{-1}J = \mathbb{I}$, we obtain the following relations,

$$\frac{\partial u^I}{\partial x^J}\big|_y = [N_-^{-1} R_+]^I{}_J\tag{5.23}$$

$$\frac{\partial u^I}{\partial y_J}\big|_x = -2[N_-^{-1}]^{IJ}.\tag{5.24}$$

Using (5.3), we compute

$$\begin{aligned}\frac{\partial v_I}{\partial x^J}\big|_u &= \frac{1}{2} [N_+]_{IJ}, \\ \frac{\partial v_I}{\partial u^J}\big|_x &= \frac{1}{2} [R_-]_{IJ},\end{aligned}\tag{5.25}$$

and obtain

$$\begin{aligned}\frac{\partial v_I}{\partial x^J}\big|_y &= \frac{\partial v_I}{\partial x^J}\big|_u + \frac{\partial v_I}{\partial u^K}\big|_x \frac{\partial u^K}{\partial x^J}\big|_y \\ &= \frac{1}{2} [N_+ + R_- N_-^{-1} R_+]_{IJ} .\end{aligned}\tag{5.26}$$

Observe that this relation is symmetric by virtue of (5.20), as it should.

Using these relations, we obtain

$$\begin{aligned}H_{ab} &= \begin{pmatrix} -\frac{1}{2} [N_+ + R_- N_-^{-1} R_+]_{IJ} & [N_-^{-1} R_+]^I{}_J \\ [N_-^{-1} R_+]^J{}_I & -2 [N_-^{-1}]^{IJ} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{2} [N_+ + R_- N_-^{-1} R_+]_{IJ} & [N_-^{-1} R_+]^I{}_J \\ [R_- N_-^{-1}]_I{}^J & -2 [N_-^{-1}]^{IJ} \end{pmatrix} .\end{aligned}\tag{5.27}$$

This expresses the Hessian in terms of second derivatives of the function $F(Y, \bar{Y}, \Upsilon, \bar{\Upsilon})$. In the absence of a chiral background, this reduces to the expression found in [12]. Using (5.9) we obtain the relation

$$H \omega H = \omega + \sigma \quad , \quad \sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} ,\tag{5.28}$$

where

$$\begin{aligned}A &= -\frac{1}{2} (N_+ N_-^{-1} R_+ - R_- N_-^{-1} N_+ + R_- N_-^{-1} (R_+ - R_-) N_-^{-1} R_+) = -A^T , \\ B &= (N_+ - N_-) N_-^{-1} + R_- N_-^{-1} (R_+ - R_-) N_-^{-1} , \\ C &= -B^T , \\ D &= -2 N_-^{-1} (R_+ - R_-) N_-^{-1} = -D^T .\end{aligned}\tag{5.29}$$

This yields $\sigma = -\sigma^T$, as expected, since $[H \omega H]^T = -H \omega H$. Observe that

$$\sigma = 0 \Leftrightarrow F_{I\bar{J}} = 0 .\tag{5.30}$$

The Hesse potential H is homogeneous of degree two under real rescalings of \mathcal{R} and of Υ , i.e. $H(\lambda \mathcal{R}, \lambda^w \Upsilon, \lambda^w \bar{\Upsilon}) = \lambda^2 H(\mathcal{R}, \Upsilon, \bar{\Upsilon})$. In the absence of a chiral background this yields the relation $H_a = H_{ab} \mathcal{R}_b$ which, when combined with $H \omega H = \omega$ (which follows from (5.28)), results in $(\omega H)_c H_{ca} = (\omega \mathcal{R})_a$. This in turn leads to a simplification of (5.15), namely to a doubling of the coefficient of the term $(\omega \mathcal{R})_a$.

We proceed to show that the field strength (2.27) is also expressed in terms of the quantities in (5.29). Using (5.3) we get for the field strength 2-form,

$$\mathcal{F} = - (d\bar{Y}^I \wedge dF_I + dY^I \wedge d\bar{F}_{\bar{I}}) = -2 (dx^I \wedge dy_I + du^I \wedge dv_I) .\tag{5.31}$$

We compute

$$\begin{aligned}2du^I \wedge dv_I &= \left(\frac{\partial u^I}{\partial x^J}\big|_y \frac{\partial v_I}{\partial x^K}\big|_y - (J \leftrightarrow K) \right) dx^J \wedge dx^K \\ &\quad + 2 \left(\frac{\partial u^I}{\partial x^J}\big|_y \frac{\partial v_I}{\partial y_K}\big|_x - \frac{\partial v_I}{\partial x^J}\big|_y \frac{\partial u^I}{\partial y_K}\big|_x \right) dx^J \wedge dy_K \\ &\quad + \left(\frac{\partial u^I}{\partial y_J}\big|_x \frac{\partial v_I}{\partial y_K}\big|_x - (J \leftrightarrow K) \right) dy_J \wedge dy_K ,\end{aligned}\tag{5.32}$$

and obtain

$$\mathcal{F} = -2 \left(2 \delta^K_J - C^K_J \right) dx^J \wedge dy_K - A_{JK} dx^J \wedge dx^K - D^{JK} dy_J \wedge dy_K , \quad (5.33)$$

where the matrices A, C and D are given in (5.29). When $\sigma = 0$ we get $\mathcal{F} = -4 dx^J \wedge dy_J$, as expected.

In [12] a new formulation of the local c-map was given by making use of the real formulation of special geometry in terms of the Hesse potential. It would be interesting to extend their analysis to the case with a chiral background.

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A Derivatives

The first-order derivatives of $\Delta(z, \bar{z}, \Psi, \bar{\Psi}, R)$ and of $F(Y, \bar{Y}, \Upsilon, \bar{\Upsilon}) = (Y^0)^2 \mathcal{F}(z, \bar{z}, \Psi, \bar{\Psi}, R)$ are given by

$$\begin{aligned} \Delta_i &= \mathcal{F}_i - (z^j - \bar{z}^j) \mathcal{F}_{ji} - w \Psi \mathcal{F}_{\Psi i} - R \mathcal{F}_{Ri} , \\ \Delta_{\bar{i}} &= 2\mathcal{F}_{\bar{i}} + \mathcal{F}_i - (z^j - \bar{z}^j) \mathcal{F}_{j\bar{i}} - w \Psi \mathcal{F}_{\Psi \bar{i}} - R \mathcal{F}_{R\bar{i}} , \\ \Delta_{\Psi} &= (2 - w) \mathcal{F}_{\Psi} - (z^i - \bar{z}^i) \mathcal{F}_{i\Psi} - w \Psi \mathcal{F}_{\Psi\Psi} - R \mathcal{F}_{R\Psi} , \\ \Delta_{\bar{\Psi}} &= 2\mathcal{F}_{\bar{\Psi}} - (z^i - \bar{z}^i) \mathcal{F}_{i\bar{\Psi}} - w \Psi \mathcal{F}_{\Psi\bar{\Psi}} - R \mathcal{F}_{R\bar{\Psi}} , \\ \Delta_R &= \mathcal{F}_R - (z^i - \bar{z}^i) \mathcal{F}_{iR} - w \Psi \mathcal{F}_{\Psi R} - R \mathcal{F}_{RR} , \end{aligned} \quad (A.1)$$

and by

$$\begin{aligned} F_0 &= Y^0 (2\mathcal{F} - z^i \mathcal{F}_i - w \Psi \mathcal{F}_{\Psi} - R \mathcal{F}_R) , \\ F_{\bar{0}} &= Y^0 (\mathcal{F}_R - \bar{R} \bar{z}^i \mathcal{F}_{\bar{i}} - w \bar{R} \bar{\Psi} \mathcal{F}_{\bar{\Psi}}) , \\ F_i &= Y^0 \mathcal{F}_i , \\ F_{\bar{i}} &= Y^0 \bar{R} \mathcal{F}_{\bar{i}} , \\ F_{\Upsilon} &= (Y^0)^{2-w} \mathcal{F}_{\Psi} , \\ F_{\bar{\Upsilon}} &= \bar{R}^2 (\bar{Y}^0)^{2-w} \mathcal{F}_{\bar{\Psi}} , \end{aligned} \quad (A.2)$$

respectively. The second-order derivatives of $F(Y, \bar{Y}, \Upsilon, \bar{\Upsilon})$ read,

$$\begin{aligned}
F_{00} &= 2\mathcal{F} - 2z^i \mathcal{F}_i + z^i z^j \mathcal{F}_{ij} + w(w-3) \Psi \mathcal{F}_\Psi + 2w z^i \Psi \mathcal{F}_{i\Psi} + w^2 \Psi^2 \mathcal{F}_{\Psi\Psi} \\
&\quad - 2R\mathcal{F}_R + 2z^i R\mathcal{F}_{iR} + 2wR\Psi \mathcal{F}_{R\Psi} + R^2 \mathcal{F}_{RR} \\
F_{0\bar{0}} &= \Delta_\Psi - \bar{z}^i \mathcal{F}_{i\Psi} \\
&\quad + \bar{R} (z^j \bar{z}^i \mathcal{F}_{i\bar{j}} - 2\bar{z}^i \mathcal{F}_{\bar{i}} - 2w\bar{\Psi} \mathcal{F}_{\bar{\Psi}} + wR\bar{\Psi} \mathcal{F}_{R\bar{\Psi}} + \bar{z}^i R\mathcal{F}_{iR} \\
&\quad + w z^i \bar{\Psi} \mathcal{F}_{i\bar{\Psi}} + w \bar{z}^i \Psi \mathcal{F}_{i\Psi} + w^2 \bar{\Psi} \Psi \mathcal{F}_{\bar{\Psi}\Psi}) , \\
F_{0i} &= \Delta_i - \bar{z}^j \mathcal{F}_{ji} \\
F_{0\bar{i}} &= \bar{R} (\Delta_{\bar{i}} - \mathcal{F}_{\bar{i}} - \bar{z}^j \mathcal{F}_{j\bar{i}}) \\
F_{i\bar{0}} &= \mathcal{F}_{Ri} - \bar{R} \bar{z}^j \mathcal{F}_{j\bar{i}} - w \bar{R} \bar{\Psi} \mathcal{F}_{\bar{\Psi}i} , \\
F_{0\Upsilon} &= (Y^0)^{1-w} (\Delta_\Psi - \bar{z}^i \mathcal{F}_{i\Psi}) , \\
F_{0\bar{\Upsilon}} &= \bar{R} (\bar{Y}^0)^{1-w} (\Delta_{\bar{\Psi}} - \bar{z}^i \mathcal{F}_{i\bar{\Psi}}) , \\
F_{ij} &= \mathcal{F}_{ij} , \\
F_{i\bar{j}} &= \bar{R} \mathcal{F}_{i\bar{j}} \\
F_{i\Upsilon} &= (Y^0)^{1-w} \mathcal{F}_{i\Psi} , \\
F_{i\bar{\Upsilon}} &= \bar{R} (\bar{Y}^0)^{1-w} \mathcal{F}_{i\bar{\Psi}} .
\end{aligned} \tag{A.3}$$

In (A.2) and (A.3), the derivatives on the left hand side are with respect to the fields $(Y^I, \bar{Y}^{\bar{I}}, \Upsilon, \bar{\Upsilon})$, while the derivatives on the right hand side refer to the projective coordinates (2.18) and their complex conjugates.

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